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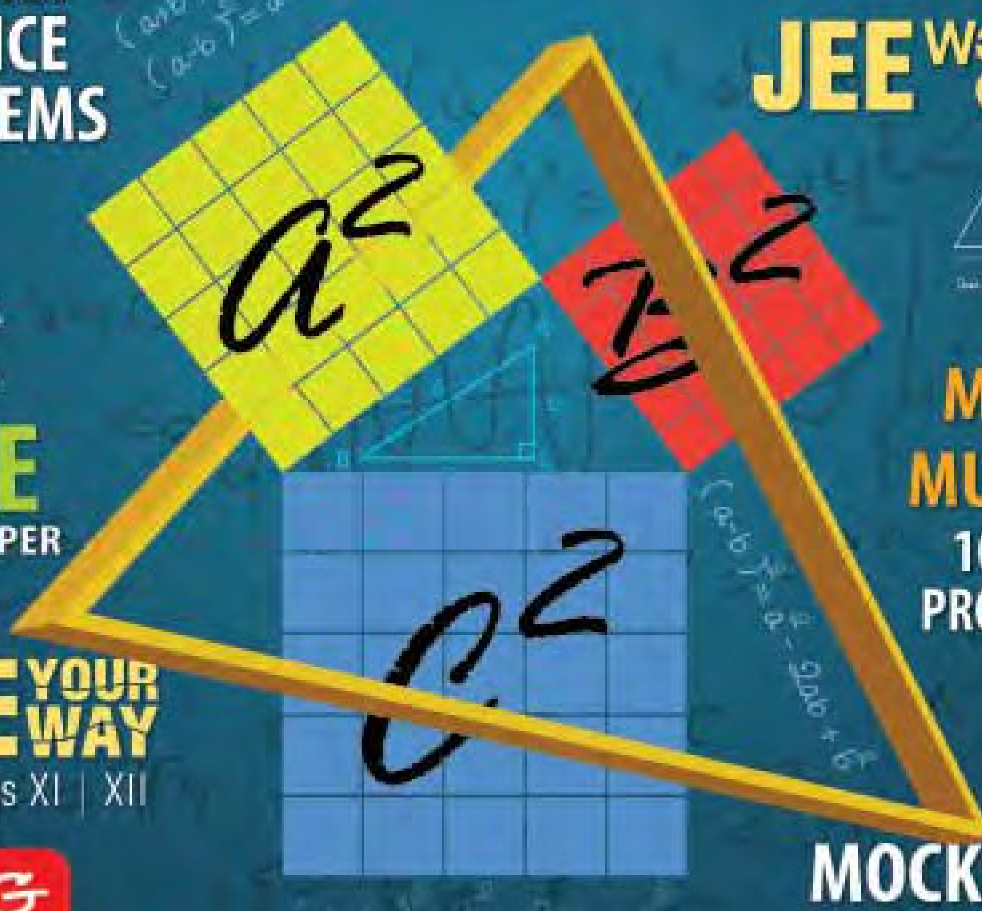

MATHS MUSING 10 GREAT PROBLEMS

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MATHEMATICS today

Vol. XXXV No. 5 May 2017

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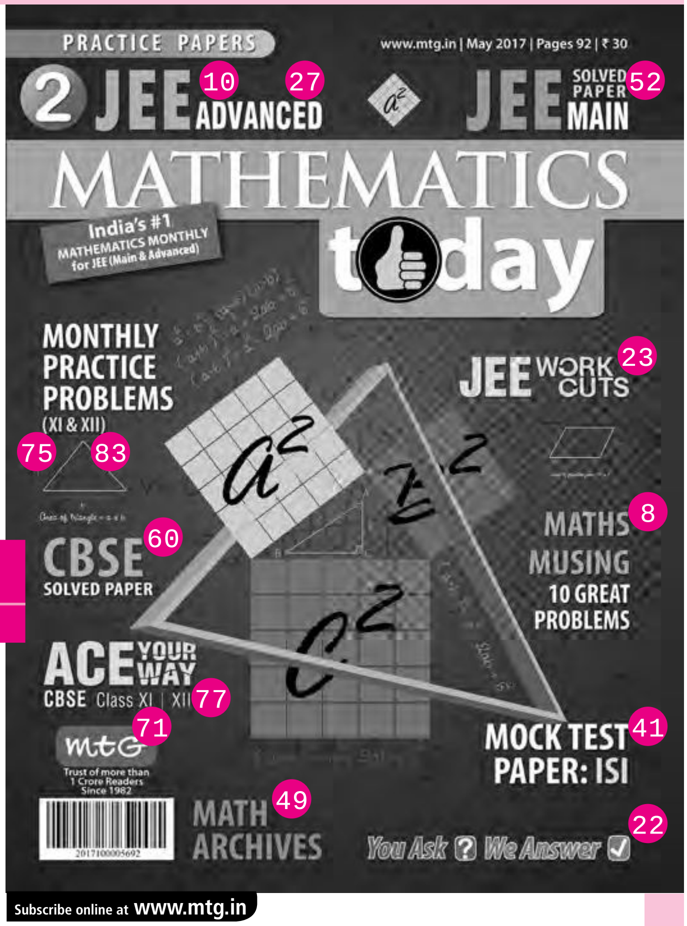

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Editor : Anil Ahlawat

MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.

During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.

Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

PROBLEM Set 173

JEE MAIN

1. A jar has 6 red marbles and 6 blue marbles. Anil picks two marbles at random, then Balu picks two of the remaining marbles at random. The probability that they get the same colour combination, irrespective of order, is

(a) $\frac{2}{7}$ (b) $\frac{4}{9}$ (c) $\frac{4}{11}$ (d) $\frac{5}{11}$

2. The locus of the middle points of chords of the circle $x^2 + y^2 = a^2$ passes through a fixed point (a, b) is

(a) $x^2 + y^2 = ax - by$ (b) $x^2 + y^2 = ax + by$

(c) $x^2 - y^2 = ax - by$ (d) none of these

3. $\int_0^{\pi/2} \frac{dx}{\cos^6 x + \sin^6 x} =$

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{3\pi}{2}$ (d) 2π

4. $\sum_{r=0}^{16} (-1)^r \binom{24}{r} =$

(a) $\binom{24}{8}$ (b) $\binom{24}{7}$ (c) $\binom{23}{8}$ (d) $\binom{23}{7}$

5. If $a_1, a_2, a_3, ...$ and $b_1, b_2, b_3, ...$ are two A.P.s, $a_1b_1 = 120$, $a_2b_2 = 143$, $a_3b_3 = 154$, then $a_8b_8 =$

(a) 29 (b) 129

(c) 229 (d) 329

JEE ADVANCED

6. The function $f(x) = \log_e[x^3 + \sqrt{x^6 + 1}]$ is an

(a) even function

(b) odd function

(c) increasing function

(d) decreasing function

COMPREHENSION

ABC is a triangle right angled at A. Points D and E are on the side AC such that $DC = 20$, $ED = 8$, $\angle ACB = \alpha$, $\angle ADB = 2\alpha$, $\angle AEB = 3\alpha$.

7. $\cos 2\alpha =$

(a) $\frac{2}{3}$ (b) $\frac{3}{4}$ (c) $\frac{4}{5}$ (d) $\frac{5}{6}$

8. $AE =$

(a) 4 (b) 5 (c) 6 (d) 7

INTEGER MATCH

9. If $a + b + c = 0$, $a^3 + b^3 + c^3 = 3$ and $a^5 + b^5 + c^5 = 10$, then $a^4 + b^4 + c^4$ is

MATRIX MATCH

10. Match the following.

	List-I		List-II
P.	When 23^{23} is divided by 53, the remainder is	1.	55
Q.	The number of integer solutions of the equation $x + y + z + u = 3$, $x \geq -2$, $y \geq -1$, $z \geq 0$, $u \geq 1$, is	2.	30
R.	The coefficient of x^2y in the expansion of $(1 + x + 2y)^5$ is	3.	56
S.	If $C_r = \binom{10}{r}$, then $\sum_{r=1}^{10} \frac{r.C_r}{C_{r-1}}$ is	4.	60

	P	Q	R	S
(a)	1	2	3	4
(b)	2	3	4	1
(c)	3	4	1	2
(d)	4	1	2	3

See Solution Set of Maths Musing 172 on page no. 51

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student speak

Shubhanan Shriniket
JEE Advanced - CG Topper

I'm Shubhanan Shriniket, a student of KCS OYP for the academic year 2015-16

I had joined KCS for my JEE 2016 main preparation. KCS has been immensely helpfull in my studies. The faculties teach concepts in such way that benefits all. Every doubt is given importance. The class is interesting for each present. Bimonthly tests are a great way to track your progress, as they are made exactly at par with JEE level. Opportunities are also provided to compare with students from other classes

[SHUBHANAN SHRINIKET]

Rishabh Kumar
JEE Main & CBSE Science
CG Topper

I am Rishabh Kumar and I prepared for JEE 2016 in KCS educate for 2 years.

KCS provided me with a sense of confidence and the environment in the classroom was very competitive. The faculty of KCS gave me constant support which helped me clear the JEE.

I would like to advise all the students preparing for JEE that they should be highly motivated and should keep themselves away from all distractions.

(RISHABH KUMAR)

Animesh Singh
JEE Adv. AIR - 555
JEE Main AIR - 386

Hello,
I am Animesh Singh and I studied in KCS in class XII.

Just as everyone needs an inspiration and willpower to achaive your target, the inspiration is the weapon that I got from KCS. Every teacher teaches but there are only a few that motivates the students and touches their hearts. And the whole faculty of KCS are from "the few".

Animesh Singh

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PRACTICE PAPER

Exam on 21st May 2017

JEE ADVANCED

*ALOK KUMAR, B.Tech, IIT Kanpur

SINGLE OPTION CORRECT TYPE

1. The number of solution(s) of equation $\sin\sin^{-1}([x]) + \cos^{-1}\cos x = 1$ (where $[\cdot]$ denotes the greatest integer function) is

(a) one (b) two
(c) three (d) none of these

2. If $P(x)$ is a polynomial with integer coefficients such that for 4 distinct integers a, b, c, d; $P(a) = P(b) = P(c) = P(d) = 3$, if $P(e) = 5$, (e is an integer) then

(a) $e = 1$ (b) $e = 3$
(c) $e = 4$ (d) no real value of e

3. A non-zero vector $\vec{a}$ is parallel to the line of intersection of the plane P_1 determined by $\hat{i}+\hat{j}$ and $\hat{i}-2\hat{j}$ and plane P_2 determined by vector $2\hat{i}+\hat{j}$ and $3\hat{i}+2\hat{k}$, then angle between $\vec{a}$ and vector $\hat{i}-2\hat{j}+2\hat{k}$ is

(a) $\pi/4$ (b) $\pi/2$
(c) $\pi/3$ (d) none of these

4. The differential equation of the system of circles touching the x-axis at origin is

(a) $(x^2-y^2)\dfrac{dy}{dx}-2xy=0$

(b) $(x^2-y^2)\dfrac{dy}{dx}+2xy=0$

(c) $(x^2+y^2)\dfrac{dy}{dx}-2xy=0$

(d) $(x^2+y^2)\dfrac{dy}{dx}+2xy=0$

5. The remainder on dividing $1234^{567} + 89^{1011}$ by 12 is

(a) 1 (b) 5
(c) 8 (d) none of these

6. If $f(x) = \sin x + \cos ax$ is periodic, then a is

(a) 2 (b) π (c) $\pi/2$ (d) $\sqrt{2}$

7. The line $y = 2x + 4$ is shifted 2 units along $+y$ axis, keeping parallel to itself and then 1 unit along $+x$ axis direction in the same manner, then equation of the line in its new position is,

(a) $y = 2x + 6$ (b) $y = 2x + 5$
(c) $y = 2x + 4$ (d) none of these

8. The number of real root(s) of the equation $x^2\tan x = 1$ lie(s) between 0 and 2π is/are

(a) 1 (b) 2 (c) 3 (d) 4

9. If $A = (p, q, r)$ and $B = (p', q', r')$ are two points on the line $\lambda x = \mu y = \gamma z$, such that $OA = 3$, $OB = 4$, then $pp' + qq' + rr'$ is equal to

(a) 7 (b) 12
(c) 5 (d) none of these

10. In ΔABC, if $b^2 + c^2 = 2a^2$, then value of $\dfrac{\cot A}{\cot B+\cot C}$ is

(a) 1/2 (b) 3/2 (c) 5/2 (d) 5/3

11. Solution of the differential the equation $y(2x^4+y)\dfrac{dy}{dx}=(1-4xy^2)x^2$ is given by

(a) $3(x^2y)^2 + y^3 - x^3 = c$

(b) $xy^2+\dfrac{y^3}{3}-\dfrac{x^3}{3}+c=0$

(c) $\dfrac{2}{5}yx^5+\dfrac{y^3}{3}=\dfrac{x^3}{3}-\dfrac{4xy^3}{3}+c$

(d) none of these

* Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91). He trains IIT and Olympiad aspirants.

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12. The value of x which satisfies the equation $2\tan^{-1}2x = \sin^{-1}\frac{4x}{1+4x^2}$ is

(a) $\left[\frac{1}{2}, \infty\right)$ (b) $\left(-\infty, -\frac{1}{2}\right]$

(c) $[-1, 1]$ (d) $\left[-\frac{1}{2}, \frac{1}{2}\right]$

13. The number of solutions of the equation $\cos^{-1}\left(\frac{1+x^2}{2x}\right) - \cos^{-1}x = \frac{\pi}{2} + \sin^{-1}x$ is given by

(a) 0 (b) 1 (c) 2 (d) 3

14. Let $f(x)$ be a continuous and differentiable function and $f(y)\, f(x + y) = f(x)\ \forall\ R$. If $f(5) = 3$ and $f'(3) = 7$, then the value of $f'(8)$ is

(a) 0 (b) 1/7

(c) 7/3 (d) 7

15. Consider 26 tangent lines to an ellipse. The lines separate the plane into several regions, some enclosed and others unbounded then numbers of unbounded regions are

(a) 50 (b) 52

(c) ${}^{26}C_2$ (d) none of these

16. A plane $2x + 3y + 5z = 1$ has point P which is at minimum distance from line joining A (1, 0, –3) and B (1, –5, 7), then distance AP is equal to

(a) $3\sqrt{5}$ (b) $2\sqrt{5}$

(c) $4\sqrt{5}$ (d) none of these

17. For the curve $x^2y^3 = c$ (where c is a constant) the portion of the tangent between the axes is divided in the ratio

(a) 3 : 5 (b) 2 : 5

(c) 3 : 2 (d) 1 : 5

18. In a triangle OAB, E is the mid point of OB and D is a point on AB such that $AD : DB = 2 : 1$. If OD and AE intersect at P, then ratio of $\frac{OP}{PD}$ is equal to

(a) 3 : 2 (b) 2 : 3

(c) 3 : 4 (d) 4 : 3

19. If α_1, α_2 and α_3 are the roots of the equation $ax^3 + bx + c = 0$, then the equation whose roots $\frac{\alpha_1^4}{\alpha_2+\alpha_3}, \frac{\alpha_2^4}{\alpha_3+\alpha_1}, \frac{\alpha_3^4}{\alpha_2+\alpha_1}$ is

(a) $a^3x^3 + 3a^2cx^2 + 3acx^2 + 3ac^2x + b^3 = 0$

(b) $a^3x^3 - 3a^2cx^2 + 3acx^2 + 3ac^2x - c^3 = 0$

(c) $a^3x^3 - 3a^2cx^2 - 3ac^2x + b^3x + c^3 = 0$

(d) None of these

20. If $a^2 + b^2 + c^2 = 1$ where $a, b, c \in R$, then the maximum value of $(4a - 3b)^2 + (5b - 4c)^2 + (3c - 5a)^2$ is

(a) 25 (b) 50

(c) 144 (d) none of these

21. If $\int_0^\infty \frac{\cos x}{x}dx = \frac{\pi}{2}$, then $\int_0^\infty \frac{1}{x}(1-\sin^2 x)^{3/2}dx$ is equal to

(a) $\pi/2$ (b) $\pi/4$

(c) $\pi/6$ (d) $3\pi/2$

22. A cubical die faces marked 1, 2, 3, ..., 6 is toaded such that the probability of throwing the number t is proportional to t^2. The probability that the number 5 has appeared given that when the die is rolled the number turned up is not even, is

(a) 1/7 (b) 3/7

(c) 5/7 (d) 2/3

23. There is a point inside an equilateral triangle ABC of side d whose distance from the vertices is 3, 4, 5. Rotate the triangle and P through 60° about C. Let A go to A' and P to P'. The area of triangle PAP' is

(a) 8 (b) 12

(c) 6 (d) none of these

ONE OR MORE THAN ONE OPTION CORRECT TYPE

24. If $f(x) = 0$ is a polynomial whose coefficients all ±1 and whose roots are all real, then the degree of $f(x)$ can be equal to

(a) 1 (b) 2

(c) 3 (d) 4

25. If $\int\sqrt{\text{cosec}x+1}\,dx = k\ fog(x) + c$, where k is a real constant, then

(a) $k = -2,\ f(x) = \cot^{-1}x, g(x) = \sqrt{\text{cosec}x - 1}$

(b) $k = -2,\ f(x) = \tan^{-1}x,\ g(x) = \sqrt{\text{cosec}x - 1}$

(c) $k = 2,\ f(x) = \tan^{-1}x,\ g(x) = \frac{\cot x}{\sqrt{\text{cosec}x - 1}}$

(d) $k = 2,\ f(x) = \cot^{-1}x,\ g(x) = \frac{\cot x}{\sqrt{\text{cosec}x + 1}}$

26. A complex number z is rotated in anticlockwise direction by an angle α and we get z' and if the same complex number z is rotated by an angle α in clockwise direction and we get z'', then
(a) z', z, z'' are in G.P.
(b) $z'^2 + z''^2 = 2z^2 \cos 2\alpha$
(c) $z' + z'' = 2z\cos\alpha$
(d) z', z, z'' are in H.P.

27. $f(x)$ is defined for $x \geq 0$ and has a continuous derivative. It satisfies $f(0) = 1$, $f'(0) = 0$ and $(1 + f(x)) f''(x) = 1 + x$. The values $f(1)$ can't take is (are)
(a) 2 (b) 1.75
(c) 1.50 (d) 1.35

28. If $z = \sec^{-1}\left(x + \frac{1}{x}\right) + \sec^{-1}\left(y + \frac{1}{y}\right)$ where $xy < 0$, then the values of z which is (are) possible
(a) $\frac{8\pi}{10}$ (b) $\frac{7\pi}{10}$
(c) $\frac{9\pi}{10}$ (d) none of these

29. Let $f(x) = [b^2 + (a-1)b + 2]x - \int(\sin^2 x + \cos^2 x)\,dx$ be an increasing function of $x \in R$ and $b \in R$, then a can take value (s)
(a) 0 (b) 1
(c) 2 (d) 4

30. The line $\frac{x-2}{3} = \frac{y-1}{2} = \frac{z-1}{-1}$ intersects the curve $x^2 - y^2 = a^2$, $z = 0$ if a is equal to
(a) 4 (b) $\sqrt{5}$
(c) –4 (d) none of these

31. π is the fundamental period of
(a) $|\sin x| + |\cos x|$
(b) $\cos(\sin x) + \cos(\cos x)$
(c) $\sin 2x + \cos 2x$
(d) none of these

32. The locus of the point of intersection of the tangents at the extremities of a chord of the circle $x^2 + y^2 = b^2$ which touches the circle $x^2 + y^2 - 2by = 0$ passes through the point
(a) $\left(0, \frac{b}{2}\right)$ (b) $(0, b)$
(c) $(b, 0)$ (d) $\left(\frac{b}{2}, 0\right)$

33. For the quadratic equation $x^2 + 2(a + 1)x + 9a - 5 = 0$ which of the following are true?
(a) If $2 < a < 5$ then roots are of opposite sign
(b) If $a < 0$, then roots are of opposite sign
(c) If $a > 7$, then both roots are negative
(d) If $2 \leq a \leq 5$ roots are unreal

COMPREHENSION TYPE

Passage for Q. No. 34 to 36

In an argand plane z_1, z_2 and z_3 are respectively the vertices of an isosceles triangle ABC with $AC = BC$ and $\angle CAB = \theta$. If z_4 is incentre of triangle. Then

34. The value of $\left(\frac{AB}{IA}\right)^2\left(\frac{AC}{AB}\right)$
(a) $\frac{(z_2 - z_1)(z_1 - z_3)}{(z_4 - z_1)^2}$ (b) $\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)}$
(c) $\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2}$ (d) none of these

35. The value $(z_4 - z_1)^2 (1 + \cos\theta)\sec\theta$ is
(a) $(z_2 - z_1)(z_3 - z_1)$ (b) $\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)}$
(c) $\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2}$ (d) $(z_2 - z_1)(z_3 - z_1)^2$

36. The value of $(z_2 - z_1)^2 \tan\theta \cdot \tan\frac{\theta}{2}$ is
(a) $(z_1 + z_2 - z_3)(z_1 + z_2 - 2z_4)$
(b) $(z_1 + z_2 - z_3)(z_1 + z_2 - z_4)$
(c) $-(z_1 + z_2 - z_3)(z_1 + z_2 - 2z_4)$
(d) none of these

MATRIX MATCH TYPE

37. Match the following :

	Column-I		Column-II
(a)	Number of solutions of the equation $\sin^{-1} x + \cos^{-1} x^2 = \frac{\pi}{2}$ is	(1)	1
(b)	The number of ordered pairs (x, y) satisfying $\frac{\sin^{-1} x}{x} + \frac{\sin^{-1} y}{y} = 2$ is	(2)	2
(c)	Number of solutions of the equation $\cos(\cos x) = \lvert\sin(\sin x)\rvert$ is	(3)	0

(d)	Number of solutions of the equation $\tan\left(x+\frac{\pi}{6}\right)=2\tan x$ is	(4)	3

38. Match the following :

	Column-I		**Column-II**
(a)	The value of k for which $\lim\limits_{x\to 1}\text{cosec}^{-1}\left(\frac{k^2}{\ln x}-\frac{k^2}{x-1}\right)$ exists is	(1)	$\left(0, \frac{2}{3}\right)$
(b)	The value of k for which $kx^2 - 2kx + 3x - 6$ is positive for exactly two integral values of x is	(2)	$[-1, 4]$
(c)	The value of k for which the point $(2k + 1, k - 1)$ is an interior point of the smaller segment of the circle $x^2 + y^2 - 2x - 4y - 4 = 0$ w.r.t. the chord $x + y - 2 = 0$ is	(3)	$\left(-\frac{3}{4}, -\frac{3}{5}\right)$
(d)	The solution of the inequality $\log_{1/5}(2x + 5) + \log_5(16 - x^2) \le 1$ is	(4)	$(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

INTEGER ANSWER TYPE

39. For $x \le 2$, then the number of possible solutions of the equation $x^3 3^{|x-2|} + 3^{x+1} = x^3 \cdot 3^{x-2} + 3^{|x-2|+3}$ is

40. The number of solutions that the equation $\sin(\cos(\sin x)) = \cos(\sin(\cos x))$ has in $\left[0, \frac{\pi}{2}\right]$ is

SOLUTIONS

1. (d) : $\sin\sin^{-1}[x] = [x]$ if $-1 \le [x] \le 1$ or $0 \le x < 2$, $\cos^{-1}\cos x = x$ if $0 \le x \le \pi$.

$\Rightarrow$ Given equation becomes $[x] + x = 1$; $0 \le x < 2$,

$\Rightarrow$ $[x] = 1 - x$; $0 \le x < 2$

$\therefore$ No solution.

2. (d) : $P(a) = P(b) = P(c) = P(d) = 3 \Rightarrow P(x) - 3$, has a, b, c, d, as its roots.

$\Rightarrow$ $P(x) - 3 = (x - a)(x - b)(x - c)(x - d)Q(x)$

($Q(x)$ has integral coefficient)

If $P(e) = 5 \Rightarrow (e - a)(e - b)(e - c)(e - d)Q(e) = 5$

This is possible only when atleast three of the 5 integers $((e - a)\ (e - b)\ (e - c)\ (e - d)\ Q(e))$ are equal to 1 or -1 $\Rightarrow$ 2 of them will be equal, which is not possible.

$\because$ a,b,c,d are distinct integers.

$\therefore$ $P(e) = 5$ is not possible.

3. (b) : Normal vector to plane $P_1 = 3\hat{k}$,

Normal vector to plane $P_2 = 2\hat{i} - 4\hat{j} - 3\hat{k}, \vec{a} = \lambda(2\hat{i} + \hat{j})$

angle between $\vec{a}$ and vector $\hat{i} - 2\hat{j} + 2\hat{k}$ is given by,

$$\cos\theta = \frac{\lambda(2\hat{i}+\hat{j})\cdot(\hat{i}-2\hat{j}+2\hat{k})}{\lambda\sqrt{5}\sqrt{9}} = 0 \Rightarrow \theta = \frac{\pi}{2}.$$

4. (a) : $(x - 0)^2 + (y - k)^2 = k^2 \Rightarrow x^2 + (y - k)^2 = k^2$,

$$2x + 2(y-k)\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y-k} \Rightarrow y - k = -\frac{xdx}{dy}$$

$$\Rightarrow k = y + \frac{xdx}{dy} \Rightarrow x^2 + \left(y - \left(y + \frac{xdx}{dy}\right)\right)^2 = \left(y + \frac{xdx}{dy}\right)^2$$

$$\Rightarrow x^2 + x^2\left(\frac{dx}{dy}\right)^2 = y^2 + x^2\left(\frac{dx}{dy}\right)^2 + \frac{2xydx}{dy}$$

$$\Rightarrow x^2 = y^2 + \frac{2xydx}{dy} \Rightarrow (x^2 - y^2)\frac{dy}{dx} - 2xy = 0$$

5. (d) : $1234^{567} \equiv 1^{567}(\text{mod } 3) = 1 \ (\text{mod } 3)$,

$89^{1011} \equiv (-1)^{1011} \ (\text{mod } 3) \equiv -1 \ (\text{mod } 3)$

then $1234^{567} + 89^{1011} \equiv 0 \ (\text{mod } 3)$

Also $1234^{567} \equiv 0 \ (\text{mod } 4)$, $89^{1011} \equiv 1 \ (\text{mod } 4)$

6. (a) : Let λ be the period of $\sin x + \cos ax$, then $\sin(\lambda + x) + \cos a(\lambda + x) = \sin x + \cos ax$ for all x. In this identity, putting $x = 0$ and $x = -\lambda$, we get

$\sin\lambda + \cos a\lambda = 1$ and $1 = -\sin\lambda + \cos a\lambda$, solving these equation, we get $\sin\lambda = 0$ and $\cos a\lambda = 1$

Hence $\lambda = n\pi$ and $a\lambda = 2m\pi$, where m, n are non-zero integers

$$\text{Hence } \frac{a\lambda}{\lambda} = \frac{2m\pi}{n\pi} \text{ or } a = \frac{2m}{n} \qquad (\text{since } \lambda \ne 0)$$

7. (c) : Any point (x_1, y_1) after shifting 2 units along $+y$ axis be $(x_1, y_1 + 2)$ and after shifting 1 units along $+x$ axis it will be $(x_1 + 1, y_1 + 2)$. Again this satisfy same equation.

8. (b)

9. (b) : $\lambda p = \mu q = \gamma r$ and $\lambda p' = \mu q' = \gamma r'$,

$$\therefore \ pp' + qq' + rr' = \frac{4}{3}(p^2 + q^2 + r^2) = \frac{4}{3}\cdot 3^2 = 12$$

10. (a) : We have,

$$\frac{\cot A}{\cot B+\cot C}=\frac{\frac{R(b^2+c^2-a^2)}{abc}}{\frac{R(a^2+c^2-b^2)}{abc}+\frac{R(a^2-c^2+b^2)}{abc}}=\frac{1}{2}$$

11. (a) : Given equation can be written as

$2x^4y\,dy + y^2\,dy + 4x^3y^2\,dx - x^2\,dx = 0$

$\Rightarrow\ 2x^2y(x^2dy + 2xy\,dx) + y^2dy - x^2dx = 0$

$\Rightarrow\ 2x^2y\,d(x^2y) + y^2\,dy - x^2dx = 0$

Integrating, we get $(x^2y)^2+\frac{y^3}{3}-\frac{x^3}{3}=c_1$

$\Rightarrow\ 3(x^2y)^2 + y^3 - x^3 = c$ (replacing $3c_1$ by c).

12. (d) : $-\frac{\pi}{2}\le 2\tan^{-1}2x\le\frac{\pi}{2},\ -\frac{\pi}{4}\le\tan^{-1}2x\le\frac{\pi}{4},$

$$-1\le 2x\le 1,-\frac{1}{2}\le x\le\frac{1}{2}$$

13. (b) : $\cos^{-1}\left(\frac{1+x^2}{2x}\right)=\frac{\pi}{2}+(\sin^{-1}x+\cos^{-1}x)$

$\Rightarrow\ \cos^{-1}\left(\frac{1+x^2}{2x}\right)=\pi$

Also, $1 + x^2 \ge 2x$ if $x > 0$ and $1 + x^2 \ge 2x$ if $x < 0$

$\Rightarrow\ \left|\frac{1+x^2}{2x}\right|\ge 1\ \Rightarrow\ \frac{1+x^2}{2x}$ can take values -1 or 1 only.

Of which $x = -1$ only satisfy the equation.

14. (c) : Put $y=5, f(5)\,f(x+5)=f(x)\ \Rightarrow\ 3f(x+5)=f(x)$.

Now differentiating w.r.t. x and putting $x = 3$, we get

$3f'(8)=f'(3)\ \Rightarrow f'(8)=\frac{7}{3}$

15. (b) : For every tangent line introduced there are two unbounded regions formed, so for 26 tangents $2 \times 26 = 52$ unbounded regions formed.

16. (b) : Let line joining AB meet plane $2x + 3y + 5z = 1$ at P.

$P=\left(\frac{\lambda+1}{\lambda+1},\frac{-5\lambda}{\lambda+1},\frac{7\lambda-3}{\lambda+1}\right)\quad\left[\frac{AP}{PB}=\lambda\right],$

$2\left(\frac{\lambda+1}{\lambda+1}\right)+3\left(\frac{-5\lambda}{\lambda+1}\right)+5\left(\frac{7\lambda-3}{\lambda+1}\right)=1$

$\Rightarrow\ 2(\lambda+1)-15\lambda+35\lambda-15=\lambda+1\ \Rightarrow\ \lambda=\frac{2}{3}$

$\Rightarrow\ P\equiv(1,-2,1)\ \Rightarrow\ AP=2\sqrt{5}$

17. (c) : Given curve is $x^2y^3 = c$

Differentiating w.r.t. x, we get

$\frac{dy}{dx}=\frac{-2y}{3x}\ \Rightarrow$ Equation of tangent at general point

(x, y) is $Y-y=-\frac{2}{3}\frac{y}{x}(X-x)$

x-intercept $=\frac{5}{2}x$, y-intercept $=\frac{5}{3}y$.

$\Rightarrow\ A\equiv\left(\frac{5}{2}x,\ 0\right)$ and $B\equiv\left(0,\ \frac{5}{3}y\right)$

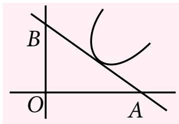
Let $AP : PB = k : 1$

$\Rightarrow\ P\equiv\left(\frac{5x}{2(k+1)},\frac{5ky}{3(k+1)}\right)$

$\Rightarrow\ \frac{5x}{2(k+1)}=x$ and $\frac{5ky}{3(k+1)}=y$

(comparing with $P\equiv(x,y)$)

$\Rightarrow\ k=\frac{3}{2}$ from both equations.



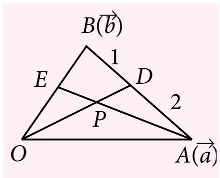
Thus P divides AB in ratio 3 : 2.

Alternative solution

For $x^my^n = c$ $(m, n > 0)$. The portion of tangent between the axes is divided in the ratio $n : m$.

Therefore required ratio is 3 : 2.

18. (a) : Let $A(\vec{a}), B(\vec{b})$



P.V. of $D=\frac{2\vec{b}+\vec{a}}{3}$

P.V. of $E=\frac{\vec{b}}{2}$

Let $\frac{OP}{PD}=t,\ \frac{AP}{PE}=\lambda,$

P.V. of $P=\frac{t(2\vec{b}+\vec{a})}{3(t+1)}=\frac{\frac{\lambda\vec{b}}{2}+\vec{a}}{\lambda+1}$

$\Rightarrow\ \frac{2t}{3(t+1)}=\frac{\lambda}{2(\lambda+1)}$ ...(1)

$\Rightarrow\ \frac{t}{3(t+1)}=\frac{1}{\lambda+1}$ ...(2)

From (1) and (2), we get

$\frac{2}{\lambda+1}=\frac{\lambda}{2(\lambda+1)}\ \Rightarrow\ \lambda=4$ Now, we get $\frac{t}{3(t+1)}=\frac{1}{5}$

$\Rightarrow\ 3t+3=5t\ \Rightarrow\ 2t=3\ \Rightarrow\ t=\frac{3}{2}$

19. (d) : $\alpha_1 + \alpha_2 + \alpha_3 = 0$

$$\Rightarrow \quad \frac{\alpha_1^4}{\alpha_2+\alpha_3} = \frac{\alpha_1^4}{-\alpha_1} = -\alpha_1^3$$

Let $\alpha_1 = x$ and $-\alpha_1^3 = y \Rightarrow y = -x^3$

$\therefore \quad x = \sqrt[3]{-y} \quad \because x$ satisfies $ax^3+bx+c=0$

$\therefore \quad a(-y)+b\sqrt[3]{(-y)}+c=0$

$\Rightarrow \quad c^3 - a^3y^3 - 3acy(c - ay) = b^3y$

$\Rightarrow \quad a^3y^3 - 3a^2cy^2 + 3ac^2y + b^3y - c^3 = 0$

20. (b) : Let $\vec{r}_1 = a\hat{i}+b\hat{j}+c\hat{k}$, $\vec{r}_2 = 3\hat{i}+4\hat{j}+5\hat{k}$

$|\vec{r}_1 \times \vec{r}_2| \le |\vec{r}_1|^2\, |\vec{r}_2|^2$ ...(1)

$$\Rightarrow \vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ 3 & 4 & 5 \end{vmatrix} = \hat{i}(5b-4c)+\hat{j}(3c-5a)+\hat{k}(4a-3b)$$

So, from (1) $(5b - 4c)^2 + (3c - 5a)^2 + (4a - 3b)^2 \le 50$.

21. (a) : $\cos 3x = 4\cos^3 x - 3\cos x$

$$\Rightarrow \quad \cos^3 x = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x,$$

Now, $\displaystyle\int_0^\infty \frac{1}{x}(1-\sin^2 x)^{3/2}\,dx = \int_0^\infty \frac{\cos^3 x}{x}dx$

$$= \frac{1}{4}\int_0^\infty \frac{\cos 3x}{x}dx + \frac{3}{4}\int_0^\infty \frac{\cos x}{x}dx$$

$$= \frac{1}{4}\int_0^\infty \frac{\cos u\,du}{u} + \frac{3}{4}\int_0^\infty \frac{\cos x}{x}dx \quad (\text{Put } u = 3x, du = 3dx)$$

$$= \frac{1}{4}\cdot\frac{\pi}{2} + \frac{3}{4}\cdot\frac{\pi}{2} = \frac{\pi}{2}$$

22. (c) : Let E_i be the event of getting i on the die

$$\sum_{i=1}^{6} P(E_i) = 1, \ \sum_{i=1}^{6} \lambda_i^2 = 1 \Rightarrow \lambda = \frac{1}{91}$$

Let A be the event of not getting an even number

$\Rightarrow \quad A = E_1 \cup E_3 \cup E_5$

$P(A) = P(E_1) + P(E_3) + P(E_5) = 35\lambda$

$\therefore$ Required probability

$$= P\left(\frac{E_5}{A}\right) = \frac{P\left(\frac{E_5}{A}\right)}{P(A)} = \frac{P(E_5)}{P(A)} = \frac{25\lambda}{35\lambda} = \frac{5}{7}$$

23. (c) : Take the triangle to be ABC and the point P. Let $PA = 3$, $PB = 4$, $PC = 5$. Rotate the triangle and P through 60° about C. Let A go to A' and P to P'. Then $CP = CP'$ and $\angle PCP' = 60°$, so PCP' is equilateral with side 5. So PAP', is a 3, 4, 5 triangle and hence $\angle PAP' = 90°$.

24. (a, b, c) : Let the equation be $a_0x^n + a_1x^{n-1} + ...a_n = 0$

$$\Rightarrow \quad \sum \alpha_i = \pm 1, \quad \sum \alpha_i\alpha_j = \pm 1 \Rightarrow \sum \alpha_i^2 = 1 \pm 2 = 3$$

where each of $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a non-zero integer. Using AM-GM inequality, $\dfrac{\sum \alpha_i^2}{n} \ge \sqrt[n]{\Pi \alpha_i^2}$

$$\Rightarrow \quad \frac{3}{n} \ge 1 \quad \therefore \quad n \le 3$$

25. (b ,d) : Let $I = \int \sqrt{\operatorname{cosec} x + 1}\, dx$

$$= \int \frac{\cot x}{\sqrt{\operatorname{cosec} x - 1}}dx \text{ put } \operatorname{cosec} x = t,\ I = -\int \frac{dt}{t\sqrt{t-1}}$$

put $t - 1 = u^2$, so that

$$I = -\int \frac{2u\,du}{u(u^2+1)} = -2\tan^{-1} u + c \ \text{ or } \ 2\cot^{-1} u + c$$

26. (a, b, c) : $z' = ze^{i\alpha}$ ...(1)

$z'' = ze^{-i\alpha}$ ...(2)

$\therefore \quad z'z'' = z^2 \Rightarrow z', z, z''$ are in G.P.

$$\Rightarrow \quad \left(\frac{z'}{z}\right)^2 + \left(\frac{z''}{z}\right)^2 = 2\cos 2\alpha$$

$\Rightarrow z'^2 + z''^2 = 2z^2 \cos 2\alpha$

$z' + z'' = 2z\cos\alpha$.

27. (a, b, c, d) : $1 + x$ is never zero, so $1 + f(x)$ is never zero. It is 1 for $x = 0$, so it is always positive.

Hence $f''(x)$ is always positive.

$f'(0) = 0$, so $f'(x) > 0$ for all $x > 0$ and hence f is strictly increasing.

So, in particular, $1 + f(x) \ge 2$ for all x.

We have $f''(x) \le \dfrac{(1+x)}{2}$

Integrating, $f'(x) \le f'(0) + \dfrac{x}{2} + \dfrac{x^2}{4} = \dfrac{x}{2} + \dfrac{x^2}{4}$

Integrating again, $f(x) \le f(0) + \dfrac{x^2}{4} + \dfrac{x^3}{12}$.

Hence $f(1) \le 1 + \dfrac{1}{4} + \dfrac{1}{12} = \dfrac{4}{3}$

28. (c) : $xy<0 \Rightarrow x+\frac{1}{x}\geq 2,\ y+\frac{1}{y}\leq -2$

or $\quad x+\frac{1}{x}\leq -2,\ y+\frac{1}{y}\geq 2$

$x+\frac{1}{x}\geq 2 \quad \Rightarrow \sec^{-1}\left(x+\frac{1}{x}\right)\in\left[\frac{\pi}{3},\frac{\pi}{2}\right)$

$y+\frac{1}{y}\leq -2 \Rightarrow \sec^{-1}\left(y+\frac{1}{y}\right)\in\left(\frac{\pi}{2},\frac{2\pi}{3}\right]$

$\Rightarrow \quad z\in\left(\frac{5\pi}{6},\frac{7\pi}{6}\right)$

29. (a, b, c) : $f'(x) = b^2 + (a-1)b + 2 - \sin^2 x - \cos^2 x$
$\Rightarrow \quad b^2 + (a-1)b + 2 - 1$ for minimum value
$\Rightarrow \quad (a-1)^2 - 4 < 0 \quad \Rightarrow a\in(-1, 3).$

30. (a, c) : For the point where the line intersects the curve, we have $z = 0$ so,
$\frac{x-2}{3}=\frac{y-1}{2}=\frac{0-1}{-1}\Rightarrow x=5$ and $y=3$
Put these values in $x^2 - y^2 = a^2$, we get $a^2 = 16$
$\Rightarrow \quad a = \pm 4$

31. (c)

32. (a, c) : If $P(h, k)$ be the point of intersection of the tangents at the extremities of the chord AB of the circle $x^2 + y^2 = b^2$. Equation of AB is $hx + ky = b^2$. This is a tangent to $x^2 + y^2 - 2by = 0$, so $\frac{h\cdot 0+k\cdot b-b^2}{\sqrt{h^2+k^2}}=\pm b$
$\Rightarrow \quad (k-b)^2 = h^2 + k^2 \Rightarrow h^2 = b(b-2k)$
$\therefore \quad$ locus of (h, k) is $x^2 = b(b-2y)$ which passes through the points $(b, 0)$ and $\left(0,\frac{b}{2}\right)$.

33. (b,c,d) : Given equation $x^2 + 2(a+1)x + 9a - 5 = 0$,
$D = 4(a+1)^2 - 4(9a-5) = 4(a-1)(a-6)$
$D \geq 0 \Rightarrow a\leq 1$ or $a \geq 6 \quad \Rightarrow$ roots are real, if $a < 0$
$\Rightarrow \quad 9a - 5 < 0 \quad \Rightarrow \quad$ products of roots is less than 0
$\Rightarrow \quad$ roots are of opposite sign, if $a > 7$ sum of roots $= -2(a+1) < 0$, product of roots > 0.

34. (c) : $\angle IAB=\frac{\theta}{2},\ \angle IAC=\frac{\theta}{2},\ \frac{z_2-z_1}{|z_2-z_1|}=\frac{z_4-z_1}{|z_4-z_1|}e^{-\frac{i\theta}{2}}$

$\frac{z_3-z_1}{|z_3-z_1|}=\frac{z_4-z_1}{|z_4-z_1|}e^{\frac{-i\theta}{2}};\ \frac{(z_2-z_1)(z_3-z_1)}{|z_2-z_1||z_3-z_1|}=\frac{(z_4-z_1)^2}{|z_4-z_1|^2}e^{0}$

$\therefore \quad \frac{(z_2-z_1)(z_3-z_1)}{(z_4-z_1)^2}=\frac{AB\cdot AC}{(IA)^2}=\left(\frac{AB}{IA}\right)^2\left(\frac{AC}{AB}\right)$

35. (a) : $\frac{(z_2-z_1)(z_3-z_1)}{(z_4-z_1)^2}=2\left(\frac{AD}{IA}\right)^2\left(\frac{AC}{AD}\right)$
(Since $AB = 2AD$)
$(z_4 - z_1)^2(1+\cos\theta)\sec\theta = (z_2 - z_1)(z_3 - z_1)$

36. (d)

37. A → 2; B → 3; C → 3; D → 3

(A) $\sin^{-1}x+\cos^{-1}x^2=\frac{\pi}{2}$

$\Rightarrow \quad \cos^{-1}x^2=\cos^{-1}x \Rightarrow x^2=x\Rightarrow x=0, 1$

(B) We have, $\frac{\sin^{-1}x}{x}+\frac{\sin^{-1}y}{y}=2$

$\frac{\sin^{-1}x}{x}$ is increasing $x\geq 0$ and decreasing for $x\leq 0$

$\Rightarrow \quad \frac{\sin^{-1}x}{x}>1$ and $\frac{\sin^{-1}y}{y}>1$

$\Rightarrow \quad \frac{\sin^{-1}x}{x}+\frac{\sin^{-1}y}{y}=2$ has no solution.

(C) $\cos x=2n\pi\pm\frac{\pi}{2}\pm\sin x \Rightarrow \cos x=2n\pi\pm\frac{\pi}{2}\pm\sin x$

$\Rightarrow \cos x\pm\sin x=2n\pi\pm\frac{\pi}{2} \Rightarrow$ no solution.

(D) $\tan\left(x+\frac{\pi}{6}\right)=2\tan x$ let $\tan x=y$

$\Rightarrow \quad 2y^2-\sqrt{3}y+1=0 \Rightarrow$ no solution.

38. A → 4; B → 3; C → 1; D → 2

(A) Limit is easily reducible to $\operatorname{cosec}^{-1}\left(\frac{k^2}{2}\right)$ by L' Hospital rule which exist if $\frac{k^2}{2}\geq 1.$

(B) $k^2x^2 + (3-2k)x - 6 = (kx+3)(kx-2)$,
$4\leq -\frac{3}{k}\leq 5\Rightarrow -\frac{3}{4}\leq k\leq -\frac{3}{5}$

(C) $(2k+1, k-1)$ is an interior point
$(2k+1)^2 + (k-1)^2 - 2(2k+1) - 4(k-1) - 4 < 0$

$\Rightarrow \quad 0<k<\frac{6}{5} \qquad$ ...(1)

Centre $(1, 2)$ and point $(2k+1, k-1)$ must lie on opposite side of chord $x + y - 2 = 0$

$\Rightarrow \quad k<\frac{2}{3} \qquad$ ...(2)

From (1) and (2), $0 < k < \frac{2}{3}$

(D) $x > -\frac{5}{2}, -4 < x < 4 \Rightarrow x \in \left(-\frac{5}{2}, 4\right), \log_5\left(\frac{16-x^2}{2x+5}\right) \le 1$

$\Rightarrow \frac{16-x^2}{2x+5} \le 5^1 \Rightarrow x \in (-\infty, -9) \cup [-1, \infty)$

$\therefore \quad x \in [-1, 4].$

39. (1) : Clearly $x = 2$ is one of the solution.

For $x < 2$, equation becomes

$x^3 3^{2-x} + 3^{x+1} = x^3 . 3^{x-2} + 3^{5-x}$

$\Rightarrow \quad 3^{2-x}(x^3 - 3^3) = 3^{x-2}(x^3 - 3^3)$

$\Rightarrow \quad (x^3 - 3^3)(3^{2-x} - 3^{x-2}) = 0$

$\Rightarrow \quad x = 2$ is the only solution.

$\therefore \quad$ Number of possible solutions for $x \le 2$ is 1.

40. (1) : Let $f(x) = \sin(\cos(\sin x)) - \cos(\sin(\cos x))$

$\Rightarrow \quad f'(x) = -\cos(\cos(\sin x))\sin(\sin x)(\cos x) - \sin(\sin(\cos x))\cos(\cos x)\sin x$

$\Rightarrow \quad f'(x) < 0 \,\forall x \in \left[0, \frac{\pi}{2}\right]$

$\Rightarrow \quad f(x)$ is decreasing in $\left[0, \frac{\pi}{2}\right]$

and $f(0) = \sin 1 - \cos(\sin 1)$ ...(1)

Now $\sin 1 - \cos(\sin 1) = \cos\left(\frac{\pi}{2} - 1\right) - \cos(\sin 1),$

$\sin 1 > \sin\frac{\pi}{4} > \frac{1}{\sqrt{2}} \therefore \frac{\pi}{2} - 1 < \sin 1 \Rightarrow \sin 1 > \frac{\pi - 2}{2}$

$\therefore \quad \sin 1 > \cos(\sin 1)$ From (1)

$\therefore \quad f(0) = \sin 1 - \cos(\sin 1) > 0, \; f\left(\frac{\pi}{2}\right) = \sin(\cos 1) - 1 < 0$

$\therefore \quad$ There will be only one root lies between $\left[0, \frac{\pi}{2}\right]$.

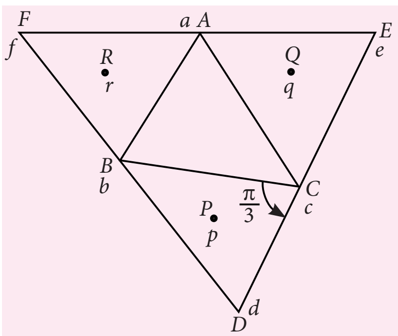
YOU ASK WE ANSWER

Do you have a question that you just can't get answered?
Use the vast expertise of our MTG team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough. The best questions and their solutions will be printed in this column each month.

1. Let ABC be a triangle. Construct equilateral triangles on the sides BC, CA, AB all externally or all internally. If P, Q, R are the centroids of these equilateral triangles, show that the triangle PQR is equilateral. *(Maithali, Delhi)*

Ans. ABC is any triangle, BCD, CAE, ABF are equilateral triangles. P, Q, R are the centroids of the equilateral triangles. Let the complex numbers $a, b, c, d, e, f, p, q, r$ represent the points A, B, C etc.

Consider the triangle BCD. Rotation about C gives



$$d - c = (b - c)\,\alpha, \text{ where } \alpha = e^{i\frac{\pi}{3}}$$

$\therefore\quad d = b\alpha + c(1 - \alpha)$

P is the centroid

$\therefore\quad 3p = b + c + d$

or $\quad 3p = (1 + \alpha)\, b + (2 - \alpha)\, c \qquad$ ...(1)

Likewise we get

$3q = (1 + \alpha)\, c + (2 - \alpha)\, a \qquad$ ...(2)

$3r = (1 + \alpha)\, a + (2 - \alpha)\, b \qquad$ ...(3)

From (1) to (3), we find

$$3(r - p) - 3\alpha\,(q - p) = (1 + \alpha)\, a + (1 - 2\alpha)\, b - (2 - \alpha)\, c - \alpha[(2 - \alpha)a - (1 + \alpha)b + (2\alpha - 1)c]$$

$$= (1 - \alpha + \alpha^2)\,(a + b - 2c) = 0$$

[since $1 - \alpha + \alpha^2 = 1 - e^{i\pi/3} + e^{2\pi i/3} = 0$]

$\therefore\quad r - p = \alpha\,(q - p)$

$\Rightarrow\quad |r - p| = |\alpha|\,|q - p| = |q - p|$.

Likewise $|p - q| = |r - q|$

$\therefore\quad |p - q| = |q - r| = |r - p| \Rightarrow \Delta PQR$ is equilateral.

2. Let the complex numbers a, b, c correspond to the points A, B, C respectively on the circle $|z| = r$. Show that the line joining A and B and the tangent to the circle at C meet at point denoted by the complex number $z = \dfrac{a^{-1} + b^{-1} - 2c^{-1}}{a^{-1}b^{-1} - c^{-2}}$.

(Sakshi Gupta, Himachal)

Ans. Let the chord AB and the tangent at C meet at P denoted by z.

$r^2 = a\bar{a} = b\bar{b} = c\bar{c}$. A, B, P are collinear.

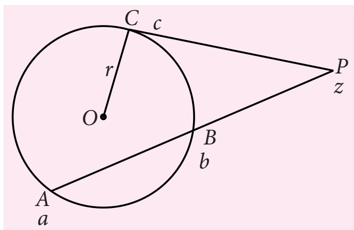
$$\frac{z-a}{b-a} = \frac{\bar{z}-\bar{a}}{\bar{b}-\bar{a}} \text{ since } \frac{z-a}{b-a} \text{ is real}$$

$\Rightarrow\quad (z - a)\,(\bar{b} - \bar{a}) - (\bar{z} - \bar{a})\,(b - a) = 0$

$$\bar{z}\,(a - b) = z\,(\bar{a} - \bar{b}) + a\bar{b} - \bar{a}b$$

$$= z\left(\frac{1}{a} - \frac{1}{b}\right)r^2 + \left(\frac{a}{b} - \frac{b}{a}\right)r^2$$

$$\therefore\quad \bar{z} = -\frac{zr^2}{ab} + \frac{(a+b)r^2}{ab} \qquad \text{...(1)}$$



$CP^2 = OP^2 - OC^2$

$\Rightarrow\quad |z - c|^2 = |z|^2 - r^2$

$\Rightarrow\quad (z - c)\,(\bar{z} - \bar{c}) = z\bar{z} - r^2$

$\Rightarrow\quad c\bar{z} = -\bar{c}\,z + 2r^2$

$$\therefore\quad \bar{z} = -\frac{zr^2}{c^2} + \frac{2r^2}{c} \qquad \text{...(2)}$$

From (1) & (2), we have $-\dfrac{z}{ab} + \dfrac{a+b}{ab} = -\dfrac{z}{c^2} + \dfrac{2}{c}$

$$\Rightarrow\ z\left(\frac{1}{ab} - \frac{1}{c^2}\right) = \frac{a+b}{ab} - \frac{2}{c} \ \Rightarrow\ z = \frac{a^{-1} + b^{-1} - 2c^{-1}}{a^{-1}b^{-1} - c^{-2}}.$$

◈◈

JEE WORK OUTS

PAPER-1

SECTION-I

STRAIGHT OBJECTIVE TYPE

[3 marks for correct answer and –1 for wrong answer]
This section contains 8 multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

1. On a railway there are 10 stations. The number of types of tickets required in order that it may be possible to book a passenger from every station to every other is:

(a) $\frac{10!}{8!}$ (b) $10!\,2!$ (c) $\frac{10!}{2!}$ (d) $\frac{10!}{8!\,2!}$

2. If the roots of $ax^2 + bx + c = 0$ are of the form $\frac{m}{m-1}$ and $\frac{m+1}{m}$ then the value of $(a + b + c)^2$ is

(a) $b^2 - 2ac$ (b) $2b^2 - ac$
(c) $b^2 - 4ac$ (d) $2(b^2 - 2ac)$

3. If A and B are two square matrices of order 3×3 which satisfy $AB = A$ and $BA = B$, then $(A + B)^7$ is

(a) $7(A+ B)$ (b) $7 \cdot I_{3 \times 3}$
(c) $64(A + B)$ (d) $128I_{3 \times 3}$

4. If a_1, a_2, a_3, a_4, a_5 are in H.P., then $a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5$ is equal to

(a) $2a_1a_5$ (b) $3a_1a_5$ (c) $8a_1a_5$ (d) $4a_1a_5$

5. The maximum value of $(\sin\alpha_1)(\sin\alpha_2)......(\sin\alpha_n)$ under the restrictions $0 \le \alpha_1, \alpha_2, ..., \alpha_n < \frac{\pi}{2}$ and $(\tan\alpha_1)(\tan\alpha_2)..........(\tan\alpha_n) = 1$ is

(a) $\frac{1}{2^n}$ (b) $\frac{1}{2n}$ (c) $\frac{1}{2^{n/2}}$ (d) 1

6. In a conference 10 speakers are to give their speaches one after another. Find the probability of the event if S_1 speaks before S_2 and S_2 speaks before S_3 and the remaining 7 speakers have no objection to speak at any number?

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{45}$ (d) $\frac{3}{10}$

7. i^i is a

(a) complex number
(b) purely imaginary number
(c) real number
(d) none of these

8. If k and K are minimum and maximum values of $\sin^{-1}x + \cos^{-1}x + \tan^{-1}x$ respectively, then

(a) $k = \frac{\pi}{4}, K = \frac{3\pi}{4}$ (b) $k = 0, K = \pi$
(c) $k = \pi/2, K = \pi$ (d) not defined

SECTION-II

MULTIPLE CORRECT ANSWER TYPE

[4 marks for correct answer and –1 for wrong answer]
This section contains 4 multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) for its answer, out of which ONE OR MORE is/are correct.

9. If $f(x)$ and $g(x)$ are functions such that $f(x + y) = f(x) \cdot g(y) + g(x) \cdot f(y)$ then $\begin{vmatrix} f(\alpha) & g(\alpha) & f(\alpha+\theta) \\ f(\beta) & g(\beta) & f(\beta+\theta) \\ f(\gamma) & g(\gamma) & f(\gamma+\theta) \end{vmatrix}$ is independent of

(a) α (b) β (c) γ (d) θ

10. Sum of the roots of the equation $(x + 1) = 2\log_2(2^x + 3) - 2\log_4(1980 - 2^{-x})$ is greater than :

(a) 2 (b) 3 (c) 4 (d) 5

11. Let $0 < P(A) < 1$, $0 < P(B) < 1$ and $P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$; then

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(a) $P(B/A) = \dfrac{P(B)}{P(A)}$

(b) $P(A^c \cup B^c) = P(A^c) + P(B^c)$

(c) $P((A \cup B)^c) = P(A^c)\ P(B^c)$

(d) $P(A/B) = P(A)$

12. The origin and roots of the equation $z^2 + pz + q = 0$ form an equilateral triangle if

(a) $p^2 = q$ (b) $p^2 = 3q$ (c) $q^2 = 3p$ (d) $q^2 = p$

SECTION-III

COMPREHENSION TYPE

[4 marks for correct answer and –1 for wrong answer] This section contains 2 paragraphs. Based upon each paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which one or more than one can be correct.

Paragraph for Question No. 13 to 15

Let p be a prime number and n be a positive integer, then exponent of p in $n!$ is denoted by $E_p(n!)$ and is given by

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots + \left[\frac{n}{p^k}\right]$$

where $p^k < n < p^{k+1}$ and $[x]$ denotes the integral part of x.

If we isolate the power of each prime contained in any number N, then N can be written as :

$N = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3}$ .... where p_1, p_2, p_3 .... primes numbers and $\alpha_1, \alpha_2, \alpha_3$ .... are natural numbers

13. The numbers of zeroes at the end of 108! is

(a) 24 (b) 25 (c) 26 (d) 21

14. The number of prime numbers among the numbers, 101! + 2, 101! + 3, 101! + 4, ... 101! + 101 is

(a) 31 (b) 29

(c) 53 (d) none of these

15. The last non-zero digit in 20! must be equal to

(a) 2 (b) 3 (c) 4 (d) 9

Paragraph for Question No. 16 to 18

The numbers 1, 3, 6, 10, 15, 21, 28, ... are called triangular numbers. Let t_n denotes the n^{th} triangular number then it can be observed that $t_1 = 1$, $t_2 = 3$, $t_n = t_{n-1} + n$. Answer the following questions :

16. t_{100} must be equal to

(a) 5050 (b) 5151

(c) 5252 (d) None of these

17. If m is the n^{th} triangular number then

(a) $n = \dfrac{\sqrt{1+8m}+1}{2}$ (b) $n = \dfrac{\sqrt{1+8m}-1}{2}$

(c) $n = \dfrac{\sqrt{1+4m}-1}{2}$ (d) None of these

18. The number of positive integers lying between t_{50} and t_{51} must be

(a) 50 (b) 51

(c) 52 (d) None of these

SECTION-IV

MATRIX MATCH TYPE

[8 marks for correct answer and no negative marking for wrong answer, for correct row 2 marks]. This section contains 2 questions. Each question contains statements given in two columns which have to be matched. Statements (A, B, C, D) in Column I have to be matched with statements (p, q, r, s, t) in Column II. The answer to these questions hare to be appropriately bubbled as illustrated in the following example. If the correct match are A – p, s and t; B – q and r; C – p and q;' and D – s and t; then the correct darkening of bubbles will look as shown.

	p	q	r	s	t
A	p	q	r	s	t
B	p	q	r	s	t
C	p	q	r	s	t
D	p	q	r	s	t

19. Match the following :

	Column I		Column II
(A)	If n be the number of ways in which 12 different books can be distributed equally among 3-persons, then $\dfrac{(4!)^4 n}{12!}$ is divisible by	(p)	4
(B)	If λ be the number of integral solutions of $x + y + z = 15$ such that $x \geq 1$, $y \geq 2$ and $z \geq 3$ then λ is divisible by	(q)	6
(C)	If λ be maximum number of points of intersection of 8 straight lines then λ is divisible by	(r)	12
(D)	A car will hold 2 persons in the front seat and 1 in the rear seat. If among 6 persons only 2 can drive, the number of ways in which car can be filled is λ then λ is divisible by	(s)	24
		(t)	5

20. If in a triangle $2a^2 + 4b^2 + c^2 = 4ab + 2ac$, then match the following :

	Column I		Column II
(A)	$\cos A$	(p)	1/4
(B)	$\cos B$	(q)	7/8
(C)	$\cos C$	(r)	0
(D)	$\sin(A - C)$	(s)	1

SECTION-I

STRAIGHT OBJECTIVE TYPE

[3 marks for correct answer and -1 for wrong answer] This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

1. The remainder when 5^{97} is divided by 52 is

(a) 2 (b) 3 (c) 5 (d) 10

2. The number of real solutions of (x, y), where $y = |\sin x|$, $y = \cos^{-1}(\cos x)$, $-2\pi \le x \le 2\pi$, is

(a) 2 (b) 1 (c) 3 (d) 4

3. In ΔABC, $\angle A = \angle C = \pi/6$ and radius of incircle is r, if radius of circumscribing circle $R = 4$ then r is equal to

(a) $4\sqrt{3}-6$ (b) $4\sqrt{3}+6$

(c) $2\sqrt{3}-2$ (d) $2\sqrt{3}+2$

4. If $2x = -1 + \sqrt{3}i$, then the value of $(1 - x^2 + x)^6 - (1 - x + x^2)^6 =$

(a) 32 (b) −64 (c) 64 (d) 0

SECTION-II

MULTIPLE CORRECT ANSWER TYPE

[4 marks for correct answer and –1 for wrong answer] This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) for its answer, out of which ONE OR MORE is/are correct.

5. Let $\{\Delta_1, \Delta_2, \Delta_3, \ldots\ldots, \Delta_k\}$ be the set of third-order determinants that can be made with the distinct nonzero real numbers $a_1, a_2, a_3, \ldots\ldots, a_9$. Then

(a) $k = 9!$ (b) $\sum_{i=1}^{k} \Delta_i = 0$

(c) at least one $\Delta_i = 0$ (d) None of these

6. The quadratic equation $x^2 - 2x - \lambda = 0, \lambda \neq 0$,

(a) cannot have a real roots if $\lambda < -1$

(b) can have a rational root if λ is a perfect square of an integer

(c) cannot have an integral root if $n^2 - 1 < \lambda < n^2 + 2n$ where $n = 0, 1, 2, 3, \ldots.$

(d) none of these

7. Given the set of four real numbers. First three numbers are in G.P. and last three numbers are in A.P. The common difference of A.P. is 6. First and fourth terms are same. According to the given data,

(a) fourth term = 4 (b) fourth term = 8

(c) sum of 4 nos. = 14 (d) sum of 4 nos. = 16

8. If $x + y + z = 5$ and $xy + yz + zx = 3$; $x, y, z \in R$ then for x :

(a) largest value $= \frac{13}{3}$ (b) largest value $= \frac{10}{3}$

(c) least value = −1 (d) least value = 5

9. In the expansion of $\left(\sqrt[3]{4} + \frac{1}{\sqrt[4]{6}}\right)^{20}$,

(a) the number of irrational terms = 19

(b) middle term is irrational

(c) the number of irrational terms = 15

(d) 9^{th} term is rational

SECTION-III

MATRIX MATCH TYPE

[8 marks for correct answer and no negative marking for wrong answer and each row 2 marks] This section contains 3 questions. Each question contains statements given in two columns which have to be matched. Statements (A, B, C, D) in Column I have to be matched with statements (p, q, r, s) in Column II. The answers to these questions have to be appropriately bubbled as illustrated in the following example. If the correct match are A – p, s and t; B – q and r; C – p and q;' and D – s and t; then the correct darkening of bubbles will look as shown.

	p	q	r	s	t
A	●	○	○	●	●
B	○	●	●	○	○
C	●	●	○	○	○
D	○	○	○	●	●

10. Match the following :

	Column I		Column II
(A)	Any two small squares on a chess board are chosen at random. Probability that they have a common side is	(p)	$\frac{1}{10}$
(B)	One of ten keys opens the door. If we try the keys one after another, then the probability that the door is opened on the tenth attempt is	(q)	0
(C)	The probability of obtaining no head in an infinite sequence of independent tosses of a coin is	(r)	$\frac{7}{144}$
(D)	Two squares are chosen at random from the small squares drawn on a chess board. The chance that the two squares chosen have exactly one corner in common is	(s)	$\frac{1}{18}$

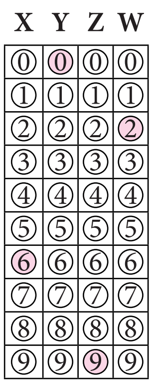
11. Match the following :

	Column I		Column II
(A)	M_r is defined as : $M_r = \begin{bmatrix} r-1 & \dfrac{1}{r} \\ 1 & \dfrac{1}{(r-1)^2} \end{bmatrix}$ and $\lvert M_r \rvert$ is the corresponding determinant value of M_r. Then $\lim_{n\to\infty}(\lvert M_2\rvert + \lvert M_3\rvert + ... + \lvert M_n\rvert) =$	(p)	-1
(B)	If $\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix} = \begin{vmatrix} 0 & \cos\alpha & \cos\beta \\ \cos\alpha & 0 & \cos\gamma \\ \cos\beta & \cos\gamma & 0 \end{vmatrix}$ then $\sin^2\alpha + \sin^2\beta + \sin^2\gamma =$	(q)	4
(C)	If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and the vectors $(1, a, a^2)$, $(1, b, b^2)$, $(1, c, c^2)$ are noncoplanar, then abc =	(r)	2
(D)	If $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $A^4 = -\lambda I$, then λ equals	(s)	1

SECTION-IV

INTEGER ANSWER TYPE

[4 marks for correct answer and –1 for wrong answer] This section contains 8 questions. The answer to each of the questions is a single-digit integer, ranging from 0 to 9. The appropriate bubbles below the respective question numbers in the ORS have to be darkened. For example, if the correct answers to question numbers X, Y, Z and W (say) are 6,0,9 and 2, respectively, then the correct darkening of bubbles will look like the following:



12. Find the number of roots of the equation $z^{10} = 1$ satisfying $\lvert \arg z \rvert < (\pi/2)$.

13. Find the number of quadratic equations which remain unchanged by squaring their roots.

14. $A(0, 0)$, $B(4, 2)$ and $C(6, 0)$ are the vertices of a triangle ABC and BD is its altitude. The line through D parallel to the side AB intersects the side BC at a point E. Find the product of areas of ΔABC and ΔBDE.

15. A coin is tossed twice and the four possible outcomes are assumed to be equally likely. If A is the event, 'both head and tail have appeared', and B be the event, 'at most one tail is observed', then $5[P(B/A)]$ is :

16. The sum of the roots of the equation

$$4\sin^3\left(\frac{\pi}{2}+x\right) - 4\cos^2 x - \cos(\pi + x) - 1 = 0$$

in the interval $[0, 4\pi]$ is $p\pi$. Find p.

17. In ΔABC, $\angle A = 30°$ and $\angle C = 105°$. Find k such that

$$k = \left(\frac{c^2 - b^2}{a^2}\right)^2.$$

18. Find the number of integral values of λ if $(\lambda, 2)$ is an interior point of ΔABC formed by $x + y = 4$, $3x - 7y = 8$, $4x - y = 31$.

19. The equation $ax^2 + bx + 6 = 0$ does not have two distinct real roots, then the least value of $3a + b + 3$ is

ANSWER KEY (PAPER-1)

1. (a) **2.** (c) **3.** (c) **4.** (d) **5.** (c)
6. (a) **7.** (c) **8.** (a) **9.** (a, b, c, d)
10. (a, b) **11.** (c, d) **12.** (b) **13.** (b) **14.** (d)
15. (c) **16.** (a) **17.** (b) **18.** (a)
19. (A)→(p),(q),(r),(s); (B)→(t); (C)→(p); (D)→(p), (t)
20. (A)→(p); (B)→(q); (C)→(p); (D)→(r)

ANSWER KEY (PAPER-2)

1. (c) **2.** (c) **3.** (a) **4.** (d) **5.** (a, b)
6. (a, c) **7.** (b, c) **8.** (a, c) **9.** (a, b, d)
10. (A)→(s); (B)→(p); (C)→(q); (D)→(r)
11. (A)→(s); (B)→(r); (C)→(p); (D)→(q)
12. (5) **13.** (4) **14.** (8) **15.** (5) **16.** (6)
17. (3) **18.** (1) **19.** (1)

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PRACTICE PAPER

JEE ADVANCED

SECTION-1

SINGLE CORRECT ANSWER TYPE

1. The number of real pairs (a, b) such that all roots of the polynomials $6x^2 - 24x - 4a$ and $x^3 + ax^2 + bx - 8$ are non-negative real numbers is/are
(a) 0 (b) 1 (c) 2 (d) 3

2. Suppose that the line segment AB has length 3 units and C is on AB with $AC = 2$ units. Equilateral triangles ACF and CBE are constructed on the same side of AB. If K is the midpoint of FC then area of ΔAKE (in sq. units) is
(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{\sqrt{3}}{4}$ (c) $2\sqrt{3}$ (d) $4\sqrt{3}$

3. The number of subsets with three elements that can be formed from the set $\{1, 2, 3, ..., 20\}$ so that 4 is a factor of the product of the three numbers on the set is
(a) 795 (b) 120 (c) 225 (d) 240

4. Let a and b be positive real numbers then
$\log(a^{10}) + {}^{10}C_1\log(a^9b) + {}^{10}C_2\log(a^8b^2) + ... + \log(b^{10}) = \log[(ab)^\lambda]$
where value of λ is
(a) 5120 (b) 2048
(c) 1024 (d) 10240

5. Given an alphabet with three letters a, b, c, the number of words of n letters, which contain an even number of a's is
(a) $(3^n + 1)$ (b) $\frac{1}{2}(3^n+1)$
(c) $3^n - 1$ (d) $\frac{1}{2}(3^n-1)$

6. Let $f(x)$ be a polynomial with integer coefficients. It is known that $f(b) - f(a) = 1$, where a and b are integers then $|a - b| =$
(a) 1 (b) 0 (c) 2 (d) 3

7. Let $S_n = {}^nC_1 - \frac{1}{2}\cdot{}^nC_2 + \frac{1}{3}\cdot{}^nC_3 .... + (-1)^{n+1}\cdot\frac{1}{n}\cdot{}^nC_n$
and $T_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$ then
(a) $S_n = T_n + 1$ (b) $S_n = 2T_n$
(c) $S_n = T_n$ (d) $S_n = 3T_n - 1$

8. If a parabola touches the lines $y = x$ and $y = -x$ at $P(3, 3)$ and $Q(1, -1)$ respectively, then
(a) equation of directrix is $y - 2x = 0$
(b) focus is $\left(\frac{-3}{5}, \frac{6}{5}\right)$
(c) directrix passes through $(1, 2)$
(d) focus is $\left(\frac{3}{5}, \frac{6}{5}\right)$

9. A straight line through a point $P(\alpha, 2)$, $(\alpha \neq 0)$ meets the ellipse $4x^2 + 9y^2 = 36$ at A and D and axes at B and C, such that PA, PB, PC, PD are in G.P. then a possible value of α is
(a) 6 (b) 8/3 (c) 2 (d) 4

10. The difference of radii of largest and smallest circle passing through the focus of parabola $y^2 = 4x$ and is contained in it is
(a) 3 (b) 3.5 (c) 4 (d) 4.5

11. 6 boys, 5 girls and 3 teachers are arranged in a line for a group photo such that boys are in ascending order, girls are in decreasing order and no two teachers are together. The number of such arrangements is
(a) $220 \times {}^{11}C_5$ (b) $3! \times 220 \times {}^{11}C_5$
(c) $3! \times {}^{11}C_6$ (d) ${}^{14}C_5 \times {}^9C_3$

12. The number of integral values of x for which the expression $\sin^{-1}\left(\frac{4x}{x^2+4}\right) - 2\tan^{-1}\left(\frac{x}{2}\right)$ is independent of x equals
(a) 4 (b) 5 (c) 6 (d) 7

By : Tapas Kr. Yogi, Mob : 9533632105.

13. The non-zero complex numbers 'a' and 'b' satisfy the condition $a \cdot 2^{|a|} + b \cdot 2^{|b|} = (a+b) \cdot 2^{|a+b|}$ then

(a) $a^2 = b^2$ (b) $a^3 = 2b^3$
(c) $a^4 = b^4$ (d) $a^6 = b^6$

14. Let ABC be an equilateral triangle of side length 1. The locus of points P in the plane of ABC such that

$$\max.\{PA, PB, PC\} = \frac{2PA \cdot PB \cdot PC}{PA \cdot PB + PB \cdot PC + PC \cdot PA - 1}$$

(a) incircle of ΔABC
(b) circumcircle of ΔABC
(c) circle through foot of altitudes of ΔABC
(d) a line bisecting two sides of the triangle

SECTION-2

COMPREHENSION TYPE

Passage-1

Consider the polynomial,

$f(x) = 4x^4 + 6x^3 + 2x^2 + 203x - (203)^2$

15. The local extrema of $f'(x)$ is

(a) positive (b) negative
(c) zero (d) data insufficient

16. The sum of the real roots of the equation $f(x) = 0$ is

(a) $-3/2$ (b) $3/2$ (c) 1 (d) -1

Passage-2

Consider the functions,

$$f(x) = \frac{\cos^2 x}{1 + \cos x + \cos^2 x} \quad \text{and}$$

$g(x) = k\tan x + (1 - k)\sin x - x$, where $k \in R$

17. $g'(x) =$

(a) $\dfrac{(1-\cos x)(k - f(x))}{f(x)}$

(b) $\dfrac{(1+\cos x)(k - f(x))}{f(x)}$

(c) $\dfrac{(1-\cos x)(k + f(x))}{f(x)}$

(d) $\dfrac{(1+\cos x)(k + f(x))}{f(x)}$

18. The range of $f(x)$ for $x \in [0, \pi/2)$ is

(a) $\left(0, \frac{1}{3}\right]$ (b) $(0, 1]$

(c) $[-1, 1]$ (d) $\left[\frac{1}{3}, \frac{4}{3}\right]$

19. The values of k for which $g(x) \geq 0, x \in [0, \pi/2]$ is

(a) $\left(0, \frac{1}{3}\right]$ (b) $\left[\frac{1}{3}, \infty\right)$

(c) $\left[-1, \frac{1}{3}\right)$ (d) $\left[\frac{1}{3}, 1\right)$

SECTION-3

INTEGER ANSWER TYPE

20. The number of 7 digit ternary sequences (*i.e.* having only digits 0, 1, 2) such that the sequence does not contain two consecutive zeros is λ, then $\left[\frac{\lambda}{1000}\right] =$ ______

([·] denotes greatest integer function)

21. A, B and C respectively take turns tossing a die. A begins, then B and then C and then again A. The probability that C will be the first one to toss a 6 is $\frac{m}{n}$, where m and n are in reduced form, then sum of digits of m is ______.

SOLUTIONS

1. **(b) :** For real roots of first equation,

$24^2 - 4(-4a) \cdot 6 \geq 0$

$\Rightarrow a \geq -6$ ... (1)

Now, let $\alpha_1, \alpha_2, \alpha_3$ be the roots of the second equation, then using A.M. $\geq$ G.M. we have

$$\frac{\alpha_1 + \alpha_2 + \alpha_3}{3} \geq \sqrt[3]{\alpha_1\alpha_2\alpha_3}$$

i.e. $\frac{-a}{3} \geq \sqrt[3]{8}$ *i.e.* $a \leq -6$ ... (2)

So, from eqns. (1) and (2), we have $a = -6$ and so $\alpha_1 = \alpha_2 = \alpha_3 = 2$ and hence $b = 12$.

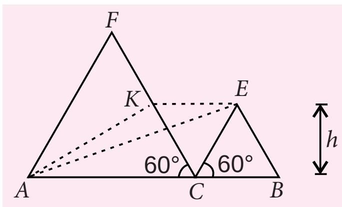
2. **(b) :** $KC = CE = \frac{1}{2}(FC) = 1$

and $\angle KCE = 60°$

So, ΔECK is equilateral

$\Rightarrow KE = 1$

and $KE \parallel ACB$ line.



So, $ar \cdot (\Delta AKE) = \frac{1}{2} \times KE \times h$

$= \frac{1}{2} \times 1 \times (1 \times \sin 60°) = \frac{\sqrt{3}}{4}$ sq. units .

3. (a) : There are in total $^{20}C_3$ subsets with 3 elements. All of these subsets have elements that when multiplied will have 4 as a factor, except in two cases.

(1) All elements are odd = $^{10}C_3$ subsets

(2) Two elements are odd and third element is even but not multiple of 4 = $^{10}C_2 \times {}^5C_1$

Hence, required subsets $= {}^{20}C_3 - {}^{10}C_3 - ({}^{10}C_2 \times {}^5C_1)$
$= 795.$

4. (a) : We have,

$\log(a^{10}) + {}^{10}C_1 \log(a^9b) + {}^{10}C_2 \log(a^8b^2) + \ldots + \log(b^{10})$
$= [\log(ab)^\lambda]$

$\Rightarrow \log(a^{10}) + \log(b^{10}) + {}^{10}C_1[\log(a^9b) + \log(ab^9)] + {}^{10}C_2$ $[\log(a^8b^2) + \log(a^2b^8)] + {}^{10}C_3[\log(a^7b^3) + \log(a^3b^7)] +$ ${}^{10}C_4[\log(a^6b^4) + \log(a^4b^6)] + {}^{10}C_5 a^5b^5 = \log[(ab)^\lambda]$

$\Rightarrow \log(ab)^{10} + {}^{10}C_1 \log(ab)^{10} + \ldots + {}^{10}C_4 \log(ab)^{10} +$

$$\frac{^{10}C_5 \log(ab)^{10}}{2} = \log[(ab)]^\lambda$$

$$\Rightarrow \log(ab)^{10}\left[1 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4 + \frac{^{10}C_5}{2}\right] = \log[(ab)^\lambda]$$

$\Rightarrow \log(ab)^{10}[1 + 10 + 45 + 120 + 210 + 126]$
$= [\log(ab)^\lambda]$

$\Rightarrow (512)\log(ab)^{10} = \log[(ab)^\lambda]$

$\therefore \quad \lambda = 5120.$

5. (b) : If there are $(2k)$ occurrences of a, these can occur in $^nC_{2k}$ places and the remaining positions can be filled in 2^{n-2k} ways. So, the net answer is

$$\sum {}^nC_{2k} \cdot 2^{n-2k} = 2^n \cdot \sum {}^nC_{2k} \cdot \left(\frac{1}{2}\right)^{2k}$$

$$= 2^n \times \frac{1}{2} \cdot \left[\left(1+\frac{1}{2}\right)^n + \left(1-\frac{1}{2}\right)^n\right] = \frac{3^n+1}{2}.$$

6. (a) : Let $f(x) = C_0 + C_1x + C_2x^2 + \ldots. + C_nx^n$ then $f(b) - f(a) = C_1(b-a) + C_2(b^2-a^2) + \ldots.$
$+ C_n(b^n - a^n)$

$= (b-a)[C_1 + C_2(b+a) + \ldots + C_n(b^{n-1} + b^{n-2}a + \ldots + a^n]$

$= (b-a) \times [\text{integer}] = 1$ (A.T.Q.)

So, $(b-a) = \pm 1.$

7. (c) : From binomial theorem, we have

$$\frac{1-(1-x)^n}{x} = {}^nC_1 - {}^nC_2x + {}^nC_3x^2 \ldots\ldots$$

Integrating on both sides within limits 0 to 1, we have

$$\int_0^1 \frac{1-y^n}{1-y}\,dy = {}^nC_1 - \frac{^nC_2x^2}{2} + \frac{^nC_3x^3}{3} + \ldots.\Bigg|_0^1$$

where $y = 1 - x$

$$i.e.\quad \int_0^1 (1 + y + y^2 + \ldots. + y^{n-1})\,dy = C_1 - \frac{C_2}{2} + \frac{C_3}{3} \ldots.$$

$$i.e.\quad 1 + \frac{1}{2} + \frac{1}{3} + \ldots. + \frac{1}{n} = C_1 - \frac{C_2}{2} + \frac{C_3}{3} \ldots.$$

i.e. $T_n = S_n.$

8. (a) : Notice that the tangents are perpendicular. Hence, their point of intersection (0, 0) lies on the directrix.

So, PQ is a focal chord.

$\Rightarrow$ Focus is foot of perpendicular from (0, 0) on focal chord PQ with eqn. $2x - y - 3 = 0$

Hence, focus is $\left(\frac{6}{5}, \frac{-3}{5}\right)$

Slope of axis $= \frac{1-0}{2-0} = \frac{1}{2}$

Directrix is $y - 0 = 2(x - 0)$ *i.e.* $y - 2x = 0.$

9. (a) : Any point on the line through $P(\alpha, 2)$ is $(\alpha + r\cos\theta, 2 + r\sin\theta)$

$$\text{So,}\quad \frac{(\alpha + r\cos\theta)^2}{9} + \frac{(2 + r\sin\theta)^2}{4} = 1$$

$$\text{gives}\quad PA \cdot PD = \frac{4\alpha^2}{4\cos^2\theta + 9\sin^2\theta}$$

$$\text{Similarly,}\quad PB \cdot PC = \frac{2\alpha}{\sin\theta\cos\theta}$$

$$\text{Hence,}\quad \frac{4\alpha^2}{4\cos^2\theta + 9\sin^2\theta} = \frac{2\alpha}{\sin\theta\cos\theta}$$

i.e. $2\alpha \sin 2\theta + 5\cos 2\theta = 13$

$\Rightarrow |\alpha| \geq 6.$

10. (b) : Minimum radius = 1/2 and for maximum radius,

$(x - r - 1)^2 + y^2 = r^2$

Solving with $y^2 = 4x$ with $D = 0$ gives $r = 4$

Hence, difference in values $= 4 - \frac{1}{2} = \frac{7}{2}$.

11. (b) : To arrange the boys and girls $= {}^{11}C_6$

To arrange the teachers $= {}^{12}C_3 \times 3!$

Hence, all arrangements $= {}^{11}C_6 \times {}^{12}C_3 \times 3!$

12. (b) : Given expression can be rewritten as

$$\sin^{-1}\left[\frac{2(x/2)}{1+(x/2)^2}\right] - 2\tan^{-1}\left[\frac{x}{2}\right]$$

$$= 2\tan^{-1}\left[\frac{x}{2}\right] - 2\tan^{-1}\left[\frac{x}{2}\right] = 0$$

= independent of x if $\left|\frac{x}{2}\right| \le 1$

i.e. $x \in [-2, 2]$. Hence integral values of $x = \pm 2, \pm 1, 0$.

13. (d) : Result : If $\alpha a + \beta b = \gamma(a + b)$ then $\alpha = \beta = \gamma$ where a, b are non-zero complex and $\alpha, \beta, \gamma \in R$.

Using this result, we have $2^{|a|} = 2^{|b|} = 2^{|a + b|}$

i.e. $|a| = |b| = |a+b|$ or $\left|\frac{a}{b}\right| = 1 = \left|\frac{a}{b}+1\right|$

i.e. $\frac{a}{b} = e^{\pm i 2\pi/3}$

Hence, $a^3 = b^3$ *i.e.* $a^6 = b^6$.

14. (b) : Let us assume that PC = max. $\{PA, PB, PC\}$ then the relation in question becomes,

$PB \cdot PC + PA \cdot PC = 1 + PA \cdot PB$

i.e., $\frac{PC}{1} = \frac{PA \cdot PB + 1 \cdot 1}{PA \cdot 1 + PC \cdot 1}$

[Ptolemy's theorem for cyclic quadrilateral]

Hence, locus of P is circumcircle of ΔABC (without the vertices A, B and C).

15. (a) : Notice that

$$\lim_{x \to \infty} f'(x) = +\infty \text{ and } \lim_{x \to -\infty} f'(x) = -\infty$$

and $f''(x) = 0$ has two real roots x_1, x_2 such that $x_1 > x_2$ hence the local minimum value (m) of $f'(x)$ is + and $x_1 \in (-1, 0)$, $m > 0$.

16. (d) : So, $f'(x) = 0$ has a unique real root.

Since, $\lim_{x \to \infty} f(x) = \infty = \lim_{x \to -\infty} f(x)$ and $f(0) < 0$

Hence, $f(x) = 0$ has exactly two real roots.

Let $a = 203$, then $f(x) = 0$ becomes

$4x^4 + 6x^3 + 2x^2 + ax - a^2 = 0$

(*i.e.* a quadratic in a)

Solving, $a = \frac{x \pm x(4x+3)}{2} = 203$

i.e. $x \pm x(4x + 3) = 406$

Solving, we have $406 = x - x(4x + 3)$ has no real roots and $406 = x + x(4x + 3)$ has sum of real roots $= -1$.

17. (a) : $g'(x) = \frac{(1-\cos x)[k(1+\cos x+\cos^2 x)-\cos^2 x)]}{\cos^2 x}$

$$= \frac{(1-\cos x)(k - f(x))}{f(x)}.$$

18. (a) : Let $t = \cos x$, $t \in (0, 1]$

So, $f(x) = h(t) = \frac{t^2}{1+t+t^2}$

$\Rightarrow$ $h'(t) > 0$ for $t \in (0, 1]$

So, $h(t)$ is increasing and $h(0) = 0$, $h(1) = 1/3$

Hence, range of $f(x)$ is $\left(0, \frac{1}{3}\right]$.

19. (b) : Notice that $k < 0$ or $k = 0$ does not give any solution.

$g(x)$ is increasing in $(0, \pi/2)$ and $g(0) = 0$

So, $g'(x) \ge 0$ $\Rightarrow$ $k \ge f(x)$ *i.e.* $k \ge 1/3$.

20. (1) : If the first digit is 0, then the next digit must be 1 or 2 and remaining $(n - 2)$ digits must not have 2 consecutive zeroes. If the first digit is 1 or 2, remaining $(n - 1)$ digits must not have 2 consecutive zeroes. So, if a_n is the required number of ways then

$a_n = 2 \cdot a_{n-1} + 2 \cdot a_{n-2}$ and $a_1 = 3$, $a_2 = 8$

gives $a_7 = 1224$.

21. (7): Let P be the probability that C wins then,

$$P = \left(\frac{5}{6}\right)^2 \times \frac{1}{6} + \left(\frac{5}{6}\right)^5 \times \frac{1}{6} + \left(\frac{5}{6}\right)^8 \times \frac{1}{6} + ...$$

$$P = \left(\frac{5}{6}\right)^2 \times \frac{1}{6}\left[1 + \left(\frac{5}{6}\right)^3 + \left(\frac{5}{6}\right)^6 + ...\right]$$

$$P = \left(\frac{5}{6}\right)^2 \times \frac{1}{6}\left[\frac{1}{1-\left(\frac{5}{6}\right)^3}\right]$$

$$P = \left(\frac{5}{6}\right)^2 \times \frac{1}{6} \times \frac{216}{91} = \frac{25}{91}$$

$\therefore$ $m = 2 + 5 = 7$.

CHALLENGING PROBLEMS

For Entrance Exams

1. The value of $\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}$ is

(a) 1 (b) $\frac{\sqrt{7}}{2}$ (c) $\frac{3\sqrt{3}}{4}$ (d) $\frac{\sqrt{15}}{4}$

2. The equation $\sin^4 x - 2\cos^2 x + a^2 = 0$ can be solved if

(a) $-\sqrt{3}\le a\le\sqrt{3}$ (b) $-\sqrt{2}\le a\le\sqrt{2}$
(c) $-1\le a\le 1$ (d) None of these

3. Given ΔABC is inscribed in the semicircle with diameter AB. The area of ΔABC equals 2/9 of the area of the semicircle. If the measure of the smallest angle in ΔABC is x, then $\sin 2x$ is equal to

(a) $\frac{\pi}{9}$ (b) $\frac{2\pi}{9}$ (c) $\frac{\pi}{18}$ (d) $\frac{\pi}{8}$

4. Set of all the values of x satisfying the inequality $\log_{x+\frac{1}{x}}\left(\log_2\frac{x-1}{x+2}\right)>0$ is

(a) $(-5, -2)$ (b) $(2, 5)$
(c) $(5, \infty)$ (d) ϕ

5. The integer value of x for which $x^2 + 19x + 92$ is square of an integer are

(a) -8 and -11 (b) -8 and 11
(c) 8 and 11 (d) None of these

6. Let a_n be the n^{th} term of the G.P. of positive numbers. Let $\sum_{n=1}^{100} a_{2n}=\alpha$ and $\sum_{n=1}^{100} a_{2n-1}=\beta$, such that $\alpha\ne\beta$, then the common ratio is

(a) $\frac{\alpha}{\beta}$ (b) $\frac{\beta}{\alpha}$ (c) $\sqrt{\frac{\alpha}{\beta}}$ (d) $\sqrt{\frac{\beta}{\alpha}}$

7. The coefficient of $x^n y^n$ in the expansion of $\{(1 + x)(1 + y)(x + y)\}^n$ is

(a) $\sum_{r=0}^{n} C_r^2$ (b) $\sum_{r=0}^{n} C_r^3$
(c) $\sum_{r+s=0}^{n} {}^nC_r\,{}^nC_s^2$ (d) None of these

8. Given 6 different toys of red colour, 5 different toys of blue colour and 4 different toys of green colour. Combination of toys that can be chosen taking at least one red and one blue toys are

(a) 31258 (b) 31248
(c) 31268 (d) None of these

9. If length of common chord of two circles $x^2 + y^2 + 8x + 1 = 0$ and $x^2 + y^2 + 2\mu y - 1 = 0$ is $2\sqrt{6}$, then value of μ are

(a) ± 2 (b) ± 3
(c) ± 4 (d) None of these

10. If two distinct chords drawn from the point (4, 4) on the parabola $y^2 = 4ax$ are bisected on the line $y = mx$, then the set of value of m is given by

(a) $\left(\frac{1-\sqrt{2}}{2},\frac{1+\sqrt{2}}{2}\right)$ (b) R
(c) $(0, \infty)$ (d) $(-2, 2)$

11. If the focal distance of an end of the minor axis of any ellipse (its axes as x and y axis respectively) is k and the distance between the foci is $2h$, then its equation is

(a) $\frac{x^2}{k^2}+\frac{y^2}{h^2}=1$ (b) $\frac{x^2}{k^2}+\frac{y^2}{k^2-h^2}=1$
(c) $\frac{x^2}{k^2}-\frac{y^2}{k^2-h^2}=1$ (d) $\frac{x^2}{k^2}+\frac{y^2}{k^2+h^2}=1$

12. If $A=\left\{x:\frac{\pi}{6}\le x\le\frac{\pi}{3}\right\}$ and $f(x) = \cos x - x(1 + x)$, then $f(A)$ is equal to

(a) $\left[\frac{1}{2}-\frac{\pi}{3}-\frac{\pi^2}{9},\frac{\sqrt{3}}{2}-\frac{\pi}{6}-\frac{\pi^2}{36}\right]$

(b) $\left[\frac{1}{2}+\frac{\pi}{3}-\frac{\pi^2}{9}, \frac{\sqrt{3}}{2}+\frac{\pi}{6}-\frac{\pi^2}{36}\right]$

(c) $\left(\frac{1}{2}-\frac{\pi}{3}-\frac{\pi^2}{9}, \frac{\sqrt{3}}{2}-\frac{\pi}{6}-\frac{\pi^2}{36}\right)$

(d) None of these

13. If the function $f(x)=\left[\frac{(x-2)^3}{a}\right]$ sin $(x - 2)$ + a cos $(x - 2)$, (where $[\cdot]$ denotes the greatest integer function) is continuous and differentiable in (4, 6), then

(a) $a \in [8, 64]$ (b) $a \in (0, 8]$

(c) $a \in [64, \infty)$ (d) None of these

14. If $f(x)=\begin{cases} 1/|x|; & |x|\geq 1 \\ ax^2+b; & -1<x<1 \end{cases}$ is differentiable $\forall\, x$, then value of a and b is

(a) $a = 1/2$, $b = -3/2$ (b) $a = -1/2$, $b = 3/2$

(c) $a = 3/2$, $b = 1/2$ (d) None of these

15. If $f''(x) > 0$ and $f'(1) = 0$ such that $g(x) = f(\cot^2 x + 2\cot x + 2)$, where $0 < x < \pi$ then the interval in which $g(x)$ is decreasing is

(a) $(0, \pi)$ (b) $\left(\frac{\pi}{2}, \pi\right)$

(c) $\left(\frac{3\pi}{4}, \pi\right)$ (d) $\left(0, \frac{3\pi}{4}\right)$

16. If $I=\int \frac{1}{x}\sqrt{\frac{1-x}{1+x}}dx$, then I equals

(a) $\log|x|+\log\left|1+\sqrt{1-x^2}\right|+\sin^{-1}\sqrt{x}+C$

(b) $\log|x|-\log\left|1-\sqrt{1-x^2}\right|+\tan^{-1}x+C$

(c) $\log|x|-\log\left|1+\sqrt{1-x^2}\right|-\sin^{-1}x+C$

(d) $\log\left|1+\sqrt{1-x^2}\right|-\log|x|+\cos^{-1}x+C$

17. If $f(x)=\tan^{-1}x+\ln\sqrt{1+x}-\ln\sqrt{1-x}$. The integral of $(1/2)\,f'(x)$ with respect to x^4 is

(a) $e^{-x^4}+c$ (b) $-\ln(1-x^4)+c$

(c) $e^{\sqrt{1-x^4}}+c$ (d) $\ln(1+x^2)+c$

18. The value of $\int_1^{16} \tan^{-1}\sqrt{\sqrt{x}-1}\,dx$ is

(a) $\frac{16\pi}{3}+2\sqrt{3}$ (b) $\frac{4}{3}\pi-2\sqrt{3}$

(c) $\frac{4}{3}\pi+2\sqrt{3}$ (d) $\frac{16}{3}\pi-2\sqrt{3}$

19. If f is a continuous function such that $\int_0^x f(t)dt \to \infty$ as $|x|\to\infty$, then for all $k \in R$, then $k^2x^2+\int_0^x f(t)dt-a=0\,(a>0)$ has

(a) all roots in $(-\infty, 0)$

(b) all roots in $(0, \infty)$

(c) odd number of roots in $(-\infty, 0)$ and odd number of roots in $(0, \infty)$

(d) None of these

20. Through any point (x, y) of a curve which passes through the origin, lines are drawn parallel to the coordinate axes. The curve, given that it divides the rectangle formed by the two lines and the axes into two areas, one of which is twice the other, represents a family of

(a) circles (b) parabolas

(c) hyperbolas (d) straight lines

21. The area bounded by the curve $y = x^4 - 2x^3 + x^2 + 3$, the axis of abscissa and two ordinates corresponding to the points of minimum of the function $y(x)$ (in sq. units) is

(a) 10/3 (b) 27/10

(c) 21/10 (d) None of these

22. Solution of the equation $y=x\frac{dy}{dx}+\frac{dx}{dy}$ represents

(a) Family of straight lines and a parabola

(b) Family of straight lines and a hyperbola

(c) Family of circles and parabola

(d) None of these

23. The differential equation of family of parabola with foci at the origin and axis along the x axes is

(a) $y\left(\frac{dy}{dx}\right)^2+2x\frac{dy}{dx}-y=0$

(b) $x\left(\frac{dy}{dx}\right)^2+2y\frac{dy}{dx}-y=0$

(c) $y\left(\frac{dy}{dx}\right)^2+2x\frac{dy}{dx}+y=0$

(d) None of these

24. If z_1, z_2 are two complex numbers such that $\left|\frac{z_1-z_2}{z_1+z_2}\right|=1$ and $iz_1 = Kz_2$, where $K\in R$, then the angle between $z_1 - z_2$ and $z_1 + z_2$ is

(a) $\tan^{-1}\left(\frac{2K}{K^2+1}\right)$ (b) $\tan^{-1}\left(\frac{K}{1-K^2}\right)$

(c) $-2\tan^{-1}K$ (d) $2\tan^{-1}K$

25. If α is a complex n^{th} root of unity and if Z_1 and Z_2 are two complex numbers, then $\sum_{r=0}^{n-1}|Z_1+\alpha^r Z_2|^2=$

(a) $n^2|Z_1 + Z_2|^2$ (b) $\left(\frac{Z_1}{n}+\frac{Z_2}{n}\right)^2$

(c) $n(|Z_1|^2 + |Z_2|^2)$ (d) $n^2(|Z_1|^2 + |Z_2|^2)$

26. Two persons A and B throw a die alternately till one of them get a "six" and wins the game. The probability of winning of B if A starts first is

(a) $\frac{6}{11}$ (b) $\frac{5}{11}$

(c) $\frac{3}{11}$ (d) None of these

27. A person throws a dice while he gets a number greater than 2. The probability that he gets a 6 in the last thrown is

(a) 2/3 (b) 1/4 (c) 1/3 (d) 1/12

28. The values of λ for which the system of equations $x + y - 3 = 0$, $(1 + \lambda)x + (2 + \lambda)y - 8 = 0$, $x - (1 + \lambda)y + (2 + \lambda) = 0$ is consistent are

(a) $-5/3$, 1 (b) $2/3$, -3

(c) $-1/3$, -3 (d) 0, 0

29. The number of values of k for which the system of equations $(k + 1)x + 8y = 4k$, $kx + (k + 3)y = 3k - 1$ has no solution is

(a) 0 (b) 1 (c) 2 (d) infinite

30. If $A=\begin{bmatrix}0&5\\0&0\end{bmatrix}$ and $f(x) = 1 + x + x^2 + ... + x^{16}$, then $f(A)$ is equal to

(a) 0 (b) $\begin{bmatrix}1&5\\0&1\end{bmatrix}$

(c) $\begin{bmatrix}1&5\\0&0\end{bmatrix}$ (d) $\begin{bmatrix}0&5\\1&1\end{bmatrix}$

31. If $|\vec{a}|=3, |\vec{b}|=4, |\vec{c}|=5$ and $\vec{a}\perp(\vec{b}+\vec{c}), \vec{b}\perp(\vec{c}+\vec{a}), \vec{c}\perp(\vec{a}+\vec{b})$ then $|\vec{a}+\vec{b}+\vec{c}|$ is equal to

(a) 50 (b) 25

(c) $5\sqrt{2}$ (d) 12

32. If lines $\frac{x-1}{2}=\frac{y-2}{x_1}=\frac{z-3}{x_2}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ lies in the same plane then for equation $x_1t^2 + (x_2 + 2)t + a = 0$, which of the options holds?

(a) $2x_1 - x_2 = 1$

(b) Sum of roots of given equation = -2

(c) $2x_1 + x_2 = -4$

(d) Sum of roots of given equation = 0

33. The variance of the first n natural numbers is

(a) $\frac{n^2-1}{12}$ (b) $\frac{n^2-1}{6}$

(c) $\frac{n^2+1}{6}$ (d) $\frac{n^2+1}{12}$

SOLUTIONS

1. (b): Let $S=\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}$ and

$C=\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}$

Now, $C + iS = \alpha + \alpha^2 + \alpha^4$ ...(i)

where, $\alpha=\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$, is complex 7^{th} root of unity

Also, $C - iS = \alpha^6 + \alpha^5 + \alpha^3$ ...(ii)

$\therefore\ \alpha^6=\bar{\alpha}, \alpha^5=\bar{\alpha}^2, \alpha^3=\bar{\alpha}^4$

Adding eqn. (i) and (ii), we get

$2C=\alpha+\alpha^2+\alpha^3+...+\alpha^6=\frac{\alpha^7-\alpha}{\alpha-1}=-1$ $(\because\ \alpha^7=1)$

$\Rightarrow\ C=-1/2$

Multiplying eqn. (i) and (ii), we get

$C^2+S^2=2 \Rightarrow S=\frac{\sqrt{7}}{2}$

2. (b): We have, $\sin^4 x - 2\cos^2 x + a^2 = 0$

or, $y^2 - 2(1-y) + a^2 = 0$ [where $\sin^2 x = y$]

$\Rightarrow y^2 + 2y + a^2 - 2 = 0$

For y to be real, discriminant ≥ 0

$\Rightarrow 4 - 4(a^2 - 2) \geq 0$

$\Rightarrow a^2 \leq 3$ ...(i)

Since, $\sin^2 x = y \Rightarrow 0 \leq y \leq 1$

$\Rightarrow 0 \leq \sqrt{3-a^2} - 1 \leq 1$

$\Rightarrow 1 \leq \sqrt{3-a^2} \leq 2 \Rightarrow 1 \leq 3 - a^2 \leq 4$

$\Rightarrow 2 - a^2 \geq 0 \Rightarrow a^2 \leq 2$

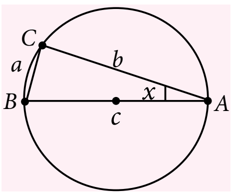
$\Rightarrow -\sqrt{2} \leq a \leq \sqrt{2}.$

3. (a): Since, $a^2 + b^2 = c^2 = 4r^2$ ...(i)

Also, $\frac{1}{2} a \cdot b = \frac{2}{9} \cdot \left(\frac{1}{2}\pi r^2\right)$

$\Rightarrow 9ab = 2\pi r^2$ ...(ii)

From eqs. (i) and (ii), we get



$\frac{a^2 + b^2}{9ab} = \frac{2}{\pi}$

$\Rightarrow \frac{a}{b} + \frac{b}{a} = \frac{18}{\pi}$

Now, $\angle BAC = x$ then $\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{18}{\pi}$

$\Rightarrow \frac{\cos^2 x + \sin^2 x}{\sin x \cdot \cos x} = \frac{18}{\pi}$

$\Rightarrow \sin x \cdot \cos x = \frac{\pi}{18} \Rightarrow \sin 2x = \frac{\pi}{9}$

4. (d): Since $\log_{x+\frac{1}{x}} \log_2 \left[\frac{x-1}{x+2}\right] > 0$

$\therefore \frac{x-1}{x+2} > 0$

$\Rightarrow x > 1, x < -2$ ...(i)

But $x + \frac{1}{x} > 0$

$\Rightarrow \frac{x^2+1}{x} > 0 \Rightarrow x > 0$ ...(ii)

From (i) and (ii), we get

$x > 1; \log_{x+\frac{1}{x}} \left(\log_2 \frac{x-1}{x+2}\right) > 0$

Take antilog both sides,

$\log_2 \frac{x-1}{x+2} > 1$

Take again antilog both sides,

$\frac{x-1}{x+2} > 2 \Rightarrow \frac{x+5}{x+2} < 0$

$\Rightarrow x \in (-5, -2)$ [But not true as $x > 1$]

5. (a): Let $x^2 + 19x + 92 = m^2$, $m \in I$

$x^2 + 19x + 92 - m^2 = 0$

$\therefore x = \frac{-19 \pm \sqrt{4m^2 - 7}}{2}$

Since, $x \in I$, so $-19 \pm \sqrt{4m^2 - 7}$ must be an even integer

$\therefore \sqrt{4m^2 - 7}$ must be an odd integer

Let $\sqrt{4m^2 - 7} = 2k + 1; k \in I$

$\therefore 4m^2 - (2k+1)^2 = 7$

$(2m - 2k - 1)(2m + 2k + 1) = 7$

Since, 7 is prime, so possible cases are as follows:

(i) $2m - 2k - 1 = 1$ and $2m + 2k + 1 = 7$ then $m = 2$

(ii) $2m - 2k - 1 = 7$ and $2m + 2k + 1 = 1$ then $m = 2$

$\therefore$ At $m = 2, x = -8, -11$

(iii) $2m - 2k - 1 = -1$ and $2m + 2k + 1 = -7$ then $m = -2$

(iv) $2m - 2k - 1 = -7$ and $2m + 2k + 1 = -1$ then $m = -2$

$\therefore$ At $m = -2, x = -8, -11$

6. (a): Let a G.P. with common ratio $= r$

Now, $\sum_{n=1}^{100} a_{2n} = a_2 + a_4 + a_6 + ... + a_{200}$

$= ar + ar^3 + ar^5 + ... + ar^{199} = \alpha$ ...(i)

Also, $\sum_{n=1}^{100} a_{2n-1} = a_1 + a_3 + a_5 + ... + a_{199}$

$= a + ar^2 + ar^4 + ... + ar^{198} = \beta$ ...(ii)

On dividing eqn. (ii) by eqn. (i), we get

$\frac{r(a + ar^2 + ar^4 + ... + ar^{198})}{(a + ar^2 + ar^4 + ... ar^{198})} = \frac{\alpha}{\beta}$

$\Rightarrow r = \frac{\alpha}{\beta}$

7. (b): We have, $\{(1+x)(1+y)(x+y)\}^n$

$= (C_0 + C_1 x + C_2 x^2 + ... + C_n x^n) \times (C_0 y^n + C_1 y^{n-1} + ... + C_{n-1} y + C_n) \times (C_0 x^n + C_1 x^{n-1} y + ... + C_{n-1} x y^{n-1} + C_n y^n)$

Coefficient of $x^n y^n$

$= C_0^3 + C_1^3 + C_2^3 + ... + C_n^3 = \sum_{r=0}^{n} C_r^3$

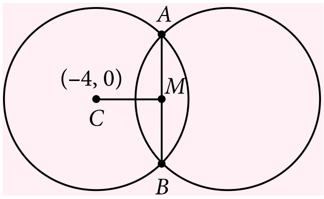
8. (b): At least one red toy can be chosen in
$= {}^6C_1 + {}^6C_2 + ... + {}^6C_6 = 2^6 - 1 = 63$ ways
At least one blue toy can be chosen in
$= {}^5C_1 + {}^5C_2 + ... + {}^5C_5 = 2^5 - 1 = 31$ ways
Green toys (with no restriction) can be selected in
$= {}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 2^4 = 16$ ways
$\therefore$ Total ways of selection $= 63 \times 31 \times 16 = 31248$

9. (b): Common chord is $S_1 - S_2 = 0$
$\Rightarrow 4x - \mu y + 1 = 0$

$\therefore AC = \sqrt{15}$

$AM = \frac{AB}{2} = \sqrt{6}$

$\therefore CM = 3$



$\therefore$ Perpendicular distance $= \frac{|-16+1|}{\sqrt{16+\mu^2}} = 3$

$\Rightarrow \mu = \pm 3$

10. (a): Any point on the line $y = mx$ can be taken as (t, mt).
Equation of the chord of parabola with (t, mt) as mid point
$y\,mt - 2(x + t) = m^2t^2 - 4t$
Since, it passes through (4, 4),
$\therefore 4mt - 2(4 + t) = m^2t^2 - 4t$
$\Rightarrow m^2t^2 - 2(2m + 1)t + 8 = 0$
For two such chords $D > 0$
$\Rightarrow (2m + 1)^2 - 8m^2 > 0$

$$\Rightarrow 4m^2 - 4m - 1 < 0 \Rightarrow \frac{1-\sqrt{2}}{2} < m < \frac{1+\sqrt{2}}{2}$$

11. (b): Let equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and e is eccentricity of ellipse.
$\therefore 2h = 2ae \Rightarrow ae = h$ ...(i)
Focal distance of one end of minor axis say $(0, b)$ is k,
$\therefore a + e(0) = k \Rightarrow a = k$ ...(ii)
From (i) and (ii),
$b^2 = a^2(1 - e^2)$
$\Rightarrow b^2 = a^2 - a^2e^2 = k^2 - h^2$

$\therefore$ Equation of ellipse is $\frac{x^2}{k^2} + \frac{y^2}{k^2 - h^2} = 1.$

12. (a): We have $f(x) = \cos x - x(1 + x)$

Since, in the interval $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$, $\cos x$ decreases and $x(1 + x)$ increases.

$\therefore f(x)$ decreases in $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$.

$$\therefore f\left(\frac{\pi}{3}\right) \le f(x) \le f\left(\frac{\pi}{6}\right); x \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$$

$$\text{Hence, } f(A) = \left[\frac{1}{2} - \frac{\pi}{3}\left(1 + \frac{\pi}{3}\right), \frac{\sqrt{3}}{2} - \frac{\pi}{6}\left(1 + \frac{\pi}{6}\right)\right]$$

$$= \left[\frac{1}{2} - \frac{\pi}{3} - \frac{\pi^2}{9}, \frac{\sqrt{3}}{2} - \frac{\pi}{6} - \frac{\pi^2}{36}\right]$$

13. (c) : We have, $x \in (4, 6)$
$\Rightarrow 2 < x - 2 < 4$

$$\Rightarrow \frac{8}{a} < \frac{(x-2)^3}{a} < \frac{64}{a}, \quad a > 0$$

For $f(x)$ to be continuous and differentiable in (4, 6), $\left[\frac{(x-2)^3}{a}\right]$ must attain a constant value for $x \in (4, 6)$.

Clearly, this is possible only when $a \ge 64$
In that case, we have
$f(x) = a\cos(x - 2)$ which is continuous and differentiable.
$\therefore a \in [64, \infty)$

14. (b): $f(x) = \begin{cases} 1/|x|; & |x| \ge 1 \\ ax^2 + b; & -1 < x < 1 \end{cases}$

$$f(x) = \begin{cases} -\frac{1}{x}; & x \le -1 \\ ax^2 + b; & -1 < x < 1 \\ \frac{1}{x}; & x \ge 1 \end{cases}$$

Since $f(x)$ is continuous at $x = 1$.
$\therefore a \cdot 1 + b = 1$ ...(i)
Also, $f(x)$ is differentiable at $x = 1$.

$\therefore 2ax = -\frac{1}{x^2}$

At $x = 1$, $2a = -1$
$\Rightarrow a = -1/2$
Using (i), we get, $b = 3/2$

15. (d): Here, $g(x) = f(\cot^2 x + 2\cot x + 2)$
$\Rightarrow g'(x) = f'(\cot^2 x + 2\cot x + 2)$
$\{-2\cot x \operatorname{cosec}^2 x - 2\operatorname{cosec}^2 x\}$
For $g(x)$ to be decreasing, $g'(x) < 0$
$\Rightarrow f'\{(\cot x + 1)^2 + 1\} \cdot (-2\operatorname{cosec}^2 x)(\cot x + 1) < 0$
$\Rightarrow f'\{(\cot x + 1)^2 + 1\} \cdot (\cot x + 1) > 0$ ...(i)
{as $f''(x) > 0 \Rightarrow f'(x)$ is increasing, then

$$f'\left\{(\cot x + 1)^2 + 1 > f'(1) = 0\ \forall x \in \left(0, \frac{3\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \pi\right)\right\}$$

Thus, equation (i) holds, if $\cot x + 1 > 0$

$$\Rightarrow \quad \cot x > -1 \ \forall x \in \left(0, \frac{3\pi}{4}\right)$$

16. (c) : $I = \int \left(\frac{1}{x}\right)\frac{1-x}{\sqrt{1-x^2}}dx = \int \frac{dx}{x\sqrt{1-x^2}} - \int \frac{dx}{\sqrt{1-x^2}}$

In the first integral, put $x = \frac{1}{t}$, we get

$$I = \int \frac{(-1/t^2)\,dt}{\frac{1}{t}\sqrt{1-\frac{1}{t^2}}} - \sin^{-1}x = -\int \frac{dt}{\sqrt{t^2-1}} - \sin^{-1}x$$

$$= -\log|t + \sqrt{t^2-1}| - \sin^{-1}x + C$$

$$= -\log\left|\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right| - \sin^{-1}x + C$$

$$= \log|x| - \log|1+\sqrt{1-x^2}| - \sin^{-1}x + C$$

17. (b): $f(x) = \tan^{-1}x + \ln\sqrt{1+x} - \ln\sqrt{1-x}$

$$f'(x) = \frac{1}{1+x^2} + \frac{1}{2(1+x)} + \frac{1}{2(1-x)}$$

$$= \frac{1}{1+x^2} + \frac{1}{1-x^2} = \frac{2}{(1-x^4)}$$

$$\Rightarrow \quad \frac{1}{2}f'(x) = \frac{1}{1-x^4} \Rightarrow \int \frac{1}{2}f'(x)dx^4 = \int \frac{dx^4}{1-x^4}$$

$= -\ln(1 - x^4) + c$

18. (d): Integrating by parts, the given integral is equal to

$$x\tan^{-1}\sqrt{\sqrt{x}-1}\Big|_1^{16} - \int_1^{16} \frac{x}{\sqrt{x}}\frac{1}{4\sqrt{x}\sqrt{\sqrt{x}-1}}dx$$

$$= \frac{16}{3}\pi - \frac{1}{4}\int_1^{16} \frac{dx}{\sqrt{\sqrt{x}-1}}$$

$$= \frac{16}{3}\pi - \frac{1}{4}\int_0^{\sqrt{3}} \frac{4t(1+t^2)}{t}dt \qquad (\text{Put } \sqrt{x} = 1+t^2)$$

$$= \frac{16}{3}\pi - \left(\sqrt{3}+\sqrt{3}\right) = \frac{16}{3}\pi - 2\sqrt{3}$$

19. (c) : Let $g(x) = k^2x^2 + \int_0^x f(t)dt - a$ ...(i)

Since, $a > 0$ and $\int_0^x f(t)dt \to \infty$ as $x \to \pm\infty$ (given)

$$\Rightarrow \quad g(0) = 0 + \int_0^0 f(t)dt - a = -a < 0$$

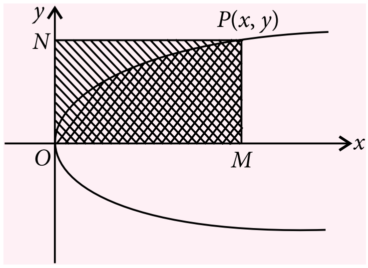
and $g(\infty) = \infty + \infty - a = \infty > 0$

Also, $g(-\infty) = \infty + \infty - a = \infty > 0$

$\Rightarrow \quad g(x)$ is continuous in $(-\infty, \infty)$.

Hence, $g(x) = 0$ has odd number of roots in $(-\infty, 0)$ and odd number of roots in $(0, \infty)$.

20. (b): Let $P(x, y)$ be the point on the curve passing through the origin $O(0, 0)$, and let PN and PM be the lines parallel to the x-axes and y-axes, respectively. If the equation of the curve is $y = y(x)$, the area POM equals $\int_0^x y\,dx$ and the area PON equals $xy - \int_0^x y dx$



Assuming that $2(POM) = PON$, we have

$$2\int_0^x y\,dx = xy - \int_0^x y\,dx \Rightarrow 3\int_0^x y\,dx = xy.$$

Differentiating both sides, we get

$$3y = x\frac{dy}{dx} + y \Rightarrow 2y = x\frac{dy}{dx} \Rightarrow \frac{dy}{y} = 2\frac{dx}{x}$$

On integrating both sides, we get

$\Rightarrow \log|y| = 2\log|x| + \log C \Rightarrow y = Cx^2$, with C being a constant.

This solutions represents a parabola. We will get a similar result if we had started instead with $2(PON) = POM$.

21. (d): Given curve is

$y = x^4 - 2x^3 + x^2 + 3$

$$\therefore \quad \frac{dy}{dx} = 4x^3 - 6x^2 + 2x$$

$$\frac{d^2y}{dx^2} = 12x^2 - 12x + 2$$

For maxima and minima, put $\frac{dy}{dx} = 0$

$\therefore \quad 4x^3 - 6x^2 + 2x = 0$

$$\Rightarrow \quad x = 0, \frac{1}{2}, 1$$

$$\therefore \quad \left.\frac{d^2y}{dx^2}\right|_{x=0} = 2, \ \left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{2}} = -1 \text{ and } \left.\frac{d^2y}{dx^2}\right|_{x=1} = 2$$

$\therefore$ Points of minima are $x = 0$ and $x = 1$

$\therefore$ Required area $= \int_0^1 (x^4 - 2x^3 + x^2 + 3)\,dx$

$$= \left[\frac{x^5}{5} - \frac{2x^4}{4} + \frac{x^3}{3} + 3x\right]_0^1$$

$$= \frac{1}{5} - \frac{2}{4} + \frac{1}{3} + 3 = \frac{1}{5} - \frac{1}{2} + \frac{1}{3} + 3 = \frac{91}{30} \text{ sq.units}$$

22. (a) : Putting $\frac{dy}{dx} = P$ in given equation, we get

$$y = xP + \frac{1}{P}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = P + x\frac{dP}{dx} - \frac{1}{P^2}\frac{dP}{dx}$$

$$\Rightarrow \quad P = P + x\frac{dP}{dx} - \frac{1}{P^2}\frac{dP}{dx} \qquad \left\{\because \frac{dy}{dx} = P\right\}$$

$$\Rightarrow \frac{dP}{dx} = 0 \text{ or } P^2 = \frac{1}{x} \quad \Rightarrow P = C \text{ or } \left(\frac{dy}{dx}\right)^2 = \frac{1}{x}$$

Put these value in given equation we get $y = Cx + \frac{1}{C}$ which is equation of family of straight lines.

and $\quad y^2 = \left(Px + \frac{1}{P}\right)^2 = P^2x^2 + 2x + \frac{1}{P^2}$

Put $P^2 = \frac{1}{x}$, we get

$y^2 = x + 2x + x = 4x$

which represents a parabola.

23. (a) : Here,

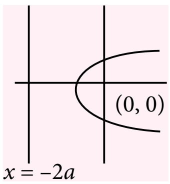
Distance from focus = distance from directrix

$x^2 + y^2 = (2a + x)^2$

$y^2 = 4a(a + x)$ ...(i)

$$2y\frac{dy}{dx} = 4a(0+1)$$

$$\Rightarrow \quad a = \frac{y}{2}\frac{dy}{dx}$$



Using (i), $y^2 = 2y\frac{dy}{dx}\left(\frac{y}{2}\frac{dy}{dx} + x\right)$

$$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$$

24. (d): $\frac{z_1 - z_2}{z_1 + z_2} = \cos\alpha + i\sin\alpha$

$$\Rightarrow \quad \frac{2z_1}{-2z_2} = \frac{\cos\alpha + i\sin\alpha + 1}{\cos\alpha + i\sin\alpha - 1}$$

$$= \frac{2\cos^2(\alpha/2) + 2i\sin(\alpha/2)\cos(\alpha/2)}{2i\sin(\alpha/2)\cos(\alpha/2) - 2\sin^2(\alpha/2)}$$

$$= \frac{2\cos(\alpha/2)[\cos(\alpha/2) + i\sin(\alpha/2)]}{2i\sin(\alpha/2)[\cos(\alpha/2) + i\sin(\alpha/2)]}$$

$$\Rightarrow \quad \frac{z_1}{z_2} = -\frac{1}{i}\cot\frac{\alpha}{2} \quad \Rightarrow \quad iz_1 = -\cot\frac{\alpha}{2}z_2$$

$\Rightarrow \; iz_1 = Kz_2 \; \Rightarrow K = -\cot\alpha/2$

$\Rightarrow \; \tan\alpha/2 = -1/K$

$$\tan\alpha = \frac{2\tan\alpha/2}{1 - \tan^2\alpha/2} \Rightarrow \frac{-2/K}{1 - 1/K^2} \Rightarrow \frac{-2K}{K^2 - 1}$$

$$\Rightarrow \quad \alpha = \tan^{-1}\left(\frac{2K}{1-K^2}\right) = 2\tan^{-1}(K)$$

25. (c) : We have $1 + \alpha + \alpha^2 + \ldots\ldots + \alpha^{n-1} = 0$.

$$\Rightarrow \quad \sum_{r=0}^{n-1} \alpha^r = 0 \qquad \text{...(i)}$$

Now, $\sum_{r=0}^{n-1} |Z_1 + \alpha^r Z_2|^2 = \sum_{r=0}^{n-1} (Z_1 + \alpha^r Z_2)(\bar{Z}_1 + \bar{\alpha}^r \bar{Z}_2)$

$$= \sum_{r=0}^{n-1} |Z_1|^2 + Z_1\bar{Z}_2 \sum_{r=0}^{n-1} \bar{\alpha}^r + \bar{Z}_1 Z_2 \sum_{r=0}^{n-1} \alpha^r + \sum_{r=0}^{n-1} |Z_2|^2 |\alpha|^{2r}$$

$= n(|Z_1|^2 + |Z_2|^2)$, [using (i) and $|\alpha| = 1$.]

26. (b): Let E = Event that A gets six

$\therefore \; P(E) = \frac{1}{6}$

F = Event that B gets six

$\therefore \; P(F) = \frac{1}{6}$

$\therefore \; P(B \text{ wins}) = P(\bar{E}\,F \text{ or } \bar{E}\,\bar{F}\,\bar{E}F \text{ or } \bar{E}\,\bar{F}\,\bar{E}\,\bar{F}\,\bar{E}F\ldots)$

(Since B can win the game in 2nd, 4th, 6th ...... throw)

$$= \left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^3\frac{1}{6} + \left(\frac{5}{6}\right)^5\frac{1}{6} + \ldots\ldots$$

$$= \frac{5}{36}\left(1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + \ldots\right) = \frac{5}{36}\left(\frac{1}{1 - \frac{25}{36}}\right) = \frac{5}{11}$$

27. (d): Let E_1 = the event that six shows when a dice is thrown

E_2 = the number less than or equal to 2 shows when a dice is thrown

$P(E_1) = 1/6$ and $P(E_2) = 2/6 = 1/3$

E = the event that six turns up in the last thrown = the event that E_1 happen in the all previous throws and E_2 happens in the last throw

$P(E) = P(E_1E_2$ or $E_1E_1E_2$ or $E_1E_1E_1E_2$.....)

$= P(E_1)\cdot P(E_2) + P(E_1)\cdot P(E_1)\cdot P(E_2) + P(E_1)\cdot P(E_1)\cdot P(E_1)\cdot P(E_2)$ +......

$$=\frac{1}{3}\cdot\frac{1}{6}+\left(\frac{1}{3}\right)^2\cdot\frac{1}{6}+\left(\frac{1}{3}\right)^3\cdot\frac{1}{6}+.....$$

$$P(E)=\frac{1}{3}\cdot\frac{1}{6}\left[\frac{1}{1-1/3}\right] \Rightarrow P(E)=\frac{1}{12}$$

28. (a) : The given system of equations will be consistent if

$$\Delta=\begin{vmatrix}1 & 1 & -3\\ 1+\lambda & 2+\lambda & -8\\ 1 & -(1+\lambda) & 2+\lambda\end{vmatrix}=0$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 + 3C_1$, we get

$$\Delta=\begin{vmatrix}1 & 0 & 0\\ 1+\lambda & 1 & -5+3\lambda\\ 1 & -2-\lambda & 5+\lambda\end{vmatrix}=0$$

$\Rightarrow\ (5 + \lambda) + (2 + \lambda)(3\lambda - 5) = 0$

$\Rightarrow\ 5 + \lambda + 6\lambda - 10 + 3\lambda^2 - 5\lambda = 0$

$\Rightarrow\ 3\lambda^2 + 2\lambda - 5 = 0 \Rightarrow (3\lambda + 5)(\lambda - 1) = 0$

$\Rightarrow\ \lambda = -5/3$ or $\lambda = 1$.

29. (b): For the system of equations having no solution, we must have

$$\frac{k+1}{k}=\frac{8}{k+3}\neq\frac{4k}{3k-1}$$

$\Rightarrow\ (k + 1)(k + 3) = 8k$ and $8(3k - 1) \neq 4k(k + 3)$

But $(k + 1)(k + 3) = 8k \Rightarrow k^2 + 4k + 3 = 8k$

or $k^2 - 4k + 3 = 0$ or $(k - 1)(k - 3) = 0 \Rightarrow k = 1$ or 3

For $k = 1$, $8(3k - 1) = 16$ and $4k(k + 3) = 16$

$\therefore\ 8(3k - 1) = 4k(k + 3)$ for $k = 1$

For $k = 3$, $8(3k - 1) = 64$ and $4k(k + 3) = 72$

i.e. $8(3k - 1) \neq 4k(k + 3)$ for $k = 3$.

Thus, there is just one value for which the given system of equations has no solution.

30. (b): $f(A) = I + A + A^2 + ... + A^{16}$

Since, $A=\begin{bmatrix}0 & 5\\ 0 & 0\end{bmatrix} \Rightarrow A^2=\begin{bmatrix}0 & 5\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 5\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

$$A^3=A^2\cdot A=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Similarly, $A^4 = A^5 = ... = A^{16} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

$$f(A)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}0 & 5\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}+....+\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$=\begin{bmatrix}1 & 5\\ 0 & 1\end{bmatrix}$$

31. (c) : $\vec{a}\perp(\vec{b}+\vec{c}) \Rightarrow \vec{a}\cdot(\vec{b}+\vec{c})=0$

$\Rightarrow\ \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=0$

Also, $\vec{b}\cdot\vec{c}+\vec{b}\cdot\vec{a}=0$ and $\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}=0$

Adding, $2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$

Now $|\vec{a}+\vec{b}+\vec{c}|^2=(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$

$=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$

$= 9 + 16 + 25 + 0 = 50$

$\therefore\ |\vec{a}+\vec{b}+\vec{c}|=5\sqrt{2}$

32. (b): Lines are coplanar if $\begin{vmatrix}1 & 1 & 1\\ 2 & x_1 & x_2\\ 3 & 4 & 5\end{vmatrix}=0$

$\Rightarrow\ 2x_1 - x_2 = 2$

Also, sum of roots $=\dfrac{-(x_2+2)}{x_1}=\dfrac{-2x_1}{x_1}=-2$

33. (a): Variance $= (\text{S.D.})^2 = \dfrac{1}{n}\Sigma x^2-\left(\dfrac{\Sigma x}{n}\right)^2$,

$$\left(\because\ \bar{x}=\frac{\Sigma x}{n}\right)$$

$$=\frac{n(n+1)(2n+1)}{6n}-\left(\frac{n(n+1)}{2n}\right)^2=\frac{n^2-1}{12}.$$

MOCK TEST PAPER

ISI

Exam on 14^{th} May 2017

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PART A

1. If $A(n)$ represents the area bounded by the curve $y = n\log_e x$, where $n \in N$ and $n > 1$, the x-axis and the lines $x = 1$ and $x = e$, then the value of $A(n) + n(A(n-1))$ is

(a) $\frac{n^2}{e+1}$ (b) $\frac{n^2}{e}$ (c) n^2 (d) $\frac{n^2}{e-1}$

2. P, Q, R and S are the points of intersection with the co-ordinate axes of the lines $px + qy = pq$ and $qx + py = pq$, then $(p, q > 0)$

(a) P, Q, R, S form a parallelogram
(b) P, Q, R, S form a rhombus
(c) P, Q, R, S are concylic
(d) none of these

3. Let $f(x)$ be a function satisfying $f'(x) = f(x)$ with $f(0) = 1$ and g be the function satisfying $f(x) + g(x) = x$, then the value of $\int_0^1 f(x)g(x)dx =$

(a) $\frac{3-e^2}{2}$ (b) $\frac{e^2-3}{2}$ (c) $\frac{e^2}{2}$ (d) $\frac{e-2}{4}$

4. If α, β, γ, δ are the smallest positive angles in ascending order of magnitude which have their sines equal to the positive quantity k, then the value of $4\sin\frac{\alpha}{2}+3\sin\frac{\beta}{2}+2\sin\frac{\gamma}{2}+\sin\frac{\delta}{2}$ is equal to

(a) $2\sqrt{1-k}$ (b) $2\sqrt{1+k}$
(c) $2\sqrt{k}$ (d) none of these

5. The greatest possible number of points of intersection of 6 straight lines and 4 circles is

(a) 55 (b) 75 (c) 45 (d) 51

6. If the points $A(z)$, $B(-z)$ and $C(z - 1)$ form an equilateral triangle, then value of z is

(a) $\frac{2\pm\sqrt{3}i}{4}$ (b) $\frac{1\pm\sqrt{3}i}{4}$
(c) $\frac{2\pm\sqrt{5}i}{4}$ (d) $\frac{2\pm 2i}{4}$

7. If α, β are two points in the domain of the function $f(x) = x + \cos x$, then which of the following is true?

(a) $\cos\alpha - \cos\beta \le \beta - \alpha$
(b) $|\cos\alpha - \cos\beta| \le |\alpha - \beta|$
(c) $\cos\beta - \cos\alpha \ge \alpha + \beta$
(d) none of these

8. If $\cos 28° + \sin 28° = k^3$, then $\cos 17°$ is equal to

(a) $\frac{k^3}{\sqrt{2}}$ (b) $-\frac{k^3}{\sqrt{2}}$
(c) $\pm\frac{k^3}{\sqrt{2}}$ (d) none of these

9. If $x^3 - 2x + 6 = 0$ has roots α, β, γ, then $\alpha^3 + \beta^3 + \gamma^3$ is equal to

(a) -18 (b) 0 (c) 2 (d) 12

10. Let $f : X \to Y$, $f(x) = \sin x + \cos x + 2\sqrt{2}$ is invertible, then $X \to Y$ is (are)

(a) $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right] \to \left[\sqrt{2}, 3\sqrt{2}\right]$
(b) $\left[-\frac{\pi}{4}, \frac{3\pi}{4}\right] \to \left[\sqrt{2}, 3\sqrt{2}\right]$
(c) $\left[\frac{3\pi}{4}, \frac{3\pi}{4}\right] \to \left[\sqrt{2}, 3\sqrt{2}\right]$
(d) $\left[-\frac{3\pi}{4}, -\frac{\pi}{4}\right] \to \left[\sqrt{2}, 3\sqrt{2}\right]$

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11. The range of values of a so that all the roots of the equation $2x^3 - 3x^2 - 12x + a = 0$ are real and distinct, belong to

(a) (7, 20) (b) (–7, 20)
(c) (–20, 7) (d) (–7, 7)

12. If α is a real root of the cubic equation $ax^3 + bx^2 + bx + a = 0$, then the value of $\lim_{x\to\frac{1}{\alpha}} \frac{\tan\left(ax^3+bx^2+bx+a\right)}{(\alpha x-1)}$ is

(a) 0 (b) $\frac{a(\alpha+1)^2}{\alpha^2}$

(c) $\frac{a(1+\alpha)}{1-\alpha}$ (d) $\frac{a(\alpha+1)^2(1-\alpha)}{\alpha^3}$

13. $\text{Min}\left[(x_1-x_2)^2+\left(5+\sqrt{1-x_1^2}-\sqrt{4x_2}\right)^2\right]$

$\forall\ x_1, x_2 \in R$ is

(a) $4\sqrt{5}+1$ (b) $4\sqrt{5}-1$
(c) $\sqrt{5}+1$ (d) $\sqrt{5}-1$

14. Number of real values of $a(a \in I)$ satisfying the equation $[\sin x]^2 + \sin x - 2a = 0$ is (where [.] denotes the greatest integer function)

(a) 0 (b) 1 (c) 2 (d) 3

15. The distances of the roots of the equation $\tan\theta_0 z^n + \tan\theta_1 z^{n-1} + \ldots + \tan\theta_n = 3$ from $z = 0$ where $\theta_0, \theta_1, \theta_2, \ldots, \theta_n \in \left[0, \frac{\pi}{4}\right]$ satisfy

(a) greater than 2/3
(b) less than 2/3
(c) greater than $|\cos\theta_1| + |\cos\theta_2| + \ldots + |\cos\theta_n|$
(d) less than $|\cos\theta_1| + |\cos\theta_2| + \ldots + |\cos\theta_n|$

16. If $x > 0,\ n \in N\ \frac{x^n}{1+x+x^2+\ldots+x^{2n}}$ is

(a) $\le \frac{1}{2n+1}$ (b) $< \frac{2}{2n+1}$

(c) $\ge \frac{1}{2n+1}$ (d) $> \frac{2}{2n+1}$

17. If $\int_2^3 \frac{x^2dx}{\sqrt{x^4-x^2+1}} = I$, then the value of

$\int_2^3 \frac{xdx}{\sqrt{1+\left(x-\frac{1}{x}\right)^2}} - \int_{1/2}^{1/3} \frac{dx}{x^3\sqrt{x^2+\frac{1}{x^2}-1}}$ is equal to

(a) $-I$ (b) 0 (c) I (d) $2I$

18. The differential equation for the family of curves $y^2 = a\sin x + b\cos x$(a, b being parameters) is

(a) $y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)+y=0$

(b) $2y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+y=0$

(c) $2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2-y=0$

(d) none of these

19. Two circles are constructed taking two sides of a triangle as diameters, then the probability of these two circles intersecting on the 3rd side of the triangle is

(a) 0 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) 1

20. If $x^2 + ax - 3x - (a + 2) = 0$ has real and distinct roots, then minimum value of $\frac{a^2+1}{a^2+2}$ is

(a) 1 (b) 0 (c) 1/2 (d) 1/4

21. If $\vec{x}$ and $\vec{y}$ be unit vectors and $|\vec{z}| = \frac{2}{\sqrt{7}}$ such that $\vec{z}+\vec{z}\times\vec{x}=\vec{y}$, then the angle θ between $\vec{x}$ and $\vec{z}$ is

(a) 30° (b) 60°
(c) 90° (d) none of these

22. Let $h(x) = x^{m/n}$ for $x \in R$ where m and n are odd numbers and $0 < m < n$, then $y = h(x)$ has

(a) no local extremums
(b) one local maximum
(c) one local minimum
(d) none of these

23. In a triangle ABC if $A \equiv (1, 2)$ and internal angle bisectors through B and C are $y = x$ and $y = -2x$. The inradius r of the ΔABC is

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{2}}$

(c) $\frac{2}{3}$ (d) none of these

24. $\lim_{x\to 0}\frac{\tan\left(\pi\sec^2 x\right)}{x\tan^{-1}x}$ is equal to

(a) $-\pi$ (b) π
(c) 0 (d) none of these

25. Let X be a set containing n elements. If two subsets A and B of X are picked at random, the probability that A and B have the same number of elements is

(a) $\frac{^{2n}C_n}{2^n}$ (b) $\frac{1}{^{2n}C_n}$

(c) $\frac{1.3...(2n-1)}{2^n \cdot n!}$ (d) $\frac{3^n}{4^n}$

PART B

1. In a triangle ABC, prove that

$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\le\frac{3}{2}$. Hence deduce that

$$\cos\frac{\pi+A}{4}\cos\frac{\pi+B}{4}\cos\frac{\pi+C}{4}\le\frac{1}{8}.$$

2. If $\frac{\cos\theta_1}{\cos\theta_2}+\frac{\sin\theta_1}{\sin\theta_2}=\frac{\cos\theta_0}{\cos\theta_2}+\frac{\sin\theta_0}{\sin\theta_2}=1$, where θ_1, and θ_0 do not differ by an even multiple of π.

Prove that $\frac{\cos\theta_1\cdot\cos\theta_0}{\cos^2\theta_2}+\frac{\sin\theta_1\cdot\sin\theta_0}{\sin^2\theta_2}=-1.$

3. Find the locus of the foot of the perpendicular, let fall from the origin upon any chord of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, which subtends a right angle at the origin.

4. Let τ_1 and τ_2 be two concentric circles. Let $A_1B_1C_1$ and $A_2B_2C_2$, be any two equilateral triangle inscribed in τ_1 and τ_2 respectively. If P_1 and P_2 be any two points on τ_1 and τ_2 respectively, show that $(P_2A_1)^2+(P_2B_1)^2+(P_2C_1)^2$ $=(P_1A_2)^2+(P_1B_2)^2+(P_1C_2)^2$.

5. In any acute angled ΔABC, $\angle A = 30°$, H is the orthocentre and M is the midpoint of BC. On the line HM, take a point T such that $HM = MT$. Show that $AT = 2BC$.

6. Two tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ having slopes m_1 and m_2 cut the axes in four concyclic points. Find the value of m_1m_2.

7. Prove that line joining the orthocentre to the centroid of a triangle formed by the focal chord of a parabola and tangents drawn at its extremities is parallel to the axis of the parabola.

8. Prove that the equation $x^6 + 2x^3 + 5 + ax^3 + a = 0$ has at the most two real roots for all values of $a \in R - \{-5\}$.

9. Find the range of the function

$$f(x)=\sin^{-1}\left(\frac{\sqrt{1+x^4}}{1+5x^{10}}\right).$$

10. If the angle A of triangle ABC is $\frac{\pi}{3}$, then prove that the vertices B, C, orthocentre, circumcentre and incentre are concyclic.

11. Find the equation of the line passing through (1, 1, 1) and perpendicular to the line of intersection of the planes $x + 2y - 4z = 0$ and $2x - y + 2z = 0$.

12. (i) Solve the equation $\cos^2x + \cos^2 2x + \cos^2 3x = 1$.
(ii) Determine all values of x in the interval $x \in [0, 2\pi]$ which satisfy the inequality $2\cos x\le|\sqrt{1+\sin 2x}-\sqrt{1-\sin 2x}|\le\sqrt{2}.$

SOLUTIONS

PART A

1. **(c):** Since, $y = n\ln(x)$, $n > 1$, $n \in N$

$$A(n)=\int_1^e n\ln(x)\,dx=n$$

$\therefore\ A(n)+n(A)(n-1)=n^2.$

2. **(a)** : If the points of intersection of two lines with co-ordinate axes be concylic, then product of intercepts on x-axis is equal to product of intercepts on y-axis by these lines. This is a geometry property. The intercepts on x-axis are b and a whose product is pq. Also the intercepts on y-axis are p, q whose product is also pq. Hence the four points are concylic.

3. **(a)** : $f'(x)=f(x)$.

Integrating, $\log f(x) = x + k$ or $f(x) = e^{x+k}$

$f(0) = 1$, $1 = e^0 \cdot e^k \Rightarrow k = 0$

$\therefore\ f(x) = e^x$

$g(x)=x-f(x)=x-e^x$

$$I=\int_0^1 e^x(x-e^x)dx=\int_0^1 e^x x\,dx-\int_0^1 e^{2x}dx$$

$$=e-(e-1)-\frac{e^2}{2}+\frac{1}{2}=\frac{3}{2}-\frac{e^2}{2}=\frac{3-e^2}{2}.$$

4. **(b)** : Given $\alpha<\beta<\gamma<\delta$

Also, $\sin\alpha = \sin\beta = \sin\gamma = \sin\delta = k$ and $\alpha, \beta, \gamma, \delta$ are smallest positive angles.

$\therefore\ \beta = \pi - \alpha, \gamma = 2\pi + \alpha, \delta = 3\pi - \alpha,$

as $\sin\beta = \sin\alpha$ and $\beta > \alpha$; $\sin\beta = \sin\gamma$ and $\gamma > \beta$; $\sin\gamma = \sin\delta$ and $\delta > \gamma$.
Putting these values in the given expression, we get

$$2\left(\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}\right)=2\sqrt{1+\sin\alpha}=2\sqrt{1+k}.$$

5. **(b)** : Two lines can meet in a point. $\therefore$ ${}^6C_2 = 15$
Line and circle meet in two points $= ({}^6C_1 \times {}^4C_1) \times 2 = 48$
Two circles meet in 2 points $= {}^4C_2 \times 2 = 12$
$\therefore$ Total number of points $= 48 + 15 + 12 = 75$.

6. **(b)** : If $z, -z$ and $z - 1$ form equilateral triangle then
$z^2 + z^2 + (z-1)^2 = -z^2 - z(z-1) + z(z-1)$
$3z^2 - 2z + 1 = -z^2 \Rightarrow 4z^2 - 2z + 1 = 0$

$$\Rightarrow z=\frac{2\pm\sqrt{4-16}}{8} \Rightarrow z=\frac{2\pm2\sqrt{3}i}{8} \Rightarrow z=\frac{1\pm\sqrt{3}i}{4}.$$

7. **(b)** : $f(x) = x = \cos x$, $f'(x) = 1 - \sin x \geq 0$
$\Rightarrow$ $f(x)$ is increasing function.
If $\alpha \geq \beta$, $\alpha + \cos\alpha \geq \beta + \cos\beta$
$\Rightarrow \cos\alpha - \cos\beta \geq \beta - \alpha$
$\Rightarrow \cos\beta - \cos\alpha \geq \alpha - \beta$ ...(1)
If $\beta \geq \alpha \Rightarrow \cos\beta + \beta \geq \cos\alpha + \alpha$
$\Rightarrow \cos\alpha - \cos\beta \leq \beta - \alpha$ ...(2)
From (1) and (2), $|\cos\alpha - \cos\beta| \leq |\alpha - \beta|$.

8. **(a)** : $\cos17° = \cos(45° - 28°)$
$= \cos45° \cos28° + \sin45° \sin28°$

$$=\frac{\cos 28°+\sin 28°}{\sqrt{2}}=\frac{k^3}{\sqrt{2}}.$$

9. **(a)** : $\alpha+\beta+\gamma=0,\ \alpha\beta+\beta\gamma+\gamma\alpha=-2,\ \alpha\beta\gamma=-6$
$\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = (\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha) = 0$
$\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = 3(-6) = -18$.

10. **(a)** : $f(x)=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)+2\sqrt{2}$

or, $f(x)=\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)+2\sqrt{2} \Rightarrow Y=\left[\sqrt{2}, 3\sqrt{2}\right]$

and $X=\left[-\frac{3\pi}{4},\frac{\pi}{4}\right]$ or $\left[\frac{\pi}{4},\frac{5\pi}{4}\right]$.

11. **(b)** : Let $f(x) = 2x^3 - 3x^2 - 12x + a$, then,
$f'(x) = 6(x^2 - x - 2) = 6(x+1)(x-2)$
So, the roots of $f'(x) = 0$ are $x = -1, 2$.
Now, $f(x) = 0$ will have all real roots if $f(-1) > 0$ and $f(2) < 0$.
$\Rightarrow -2 - 3 + 12 + a > 0$ and $16 - 12 - 24 + a < 0$
$\Rightarrow -7 < a < 20$.

12. **(d)** : Clearly, $x = -1$ and $1/\alpha$ are the other roots.

$$\Rightarrow \lim_{x\to\frac{1}{\alpha}}\frac{\tan\left[a(x+1)(x-\alpha)\left(x-\frac{1}{\alpha}\right)\right]}{(\alpha x-1)}$$

$$=\lim_{x\to\frac{1}{\alpha}}\frac{\tan\left[a(x+1)(x-\alpha)\left(x-\frac{1}{\alpha}\right)\right]}{\alpha\left(x-\frac{1}{\alpha}\right)}$$

$$=\lim_{x\to\frac{1}{\alpha}}\frac{a}{\alpha}(x+1)(x-\alpha)=\frac{a}{\alpha}\left(\frac{1}{\alpha}+1\right)\left(\frac{1}{\alpha}-1\right)$$

$$=\frac{a(\alpha+1)^2(1-\alpha)}{\alpha^3}.$$

13. **(b)** : Given expression is the shortest distance between the curves $x^2 + (y-5)^2 = 1$ and $y^2 = 4x$.
Normal to the parabola $y^2 = 4x$ is $y = mx - 2am - am^3$ passes through (0, 5) gives $m^2 + 2m + 12 = 0$, thus only 1 real value of $m = -2$.
Hence, corresponding point on the parabola is (4, 4).
Thus, required minimum distance

$$=\sqrt{4^2+8^2}-1=4\sqrt{5}-1.$$

14. **(c)**: $2a = [\sin x]^2 + \sin x \Rightarrow \sin x \in I$ (as $a \in I$)
$\Rightarrow [\sin x] = \sin x \Rightarrow 2a = \sin x(\sin x + 1)$.
Also, $\sin x$ can take the values -1, 0 and 1 only.
$\Rightarrow$ a can take only two values 0 and 1.

15. **(a)** : $3=|\tan\theta_0 z^n+\tan\theta_1 z^{n-1}+....+\tan\theta_n|$

$\Rightarrow 3\leq|\tan\theta_0||z|^n+|\tan\theta_1||z|^{n-1}+.....+|\tan\theta_n|$

$\Rightarrow 3\leq|z|^n+|z|^{n-1}+....+1$

Since, $|\tan\theta_i| \leq 1$

$\therefore$ $3 < 1 + |z| + |z^2| + ... + \infty$,

$$\Rightarrow 3<\frac{1}{1-|z|} \Rightarrow 3-3|z|<1 \Rightarrow -3|z|<1-3=-2$$

$$\Rightarrow 3|z|>2 \Rightarrow |z|>\frac{2}{3}.$$

16. **(a)** : $x+\frac{1}{x}\geq 2,....,x^n+\frac{1}{x^n}\geq 2$

On adding, $\left(x+\frac{1}{x}\right)+\left(x^2+\frac{1}{x^2}\right)+...+\left(x^n+\frac{1}{x^n}\right)\geq 2n$

$$\Rightarrow \left(\frac{1}{x^n}+\frac{1}{x^{n-1}}+....+\frac{1}{x}\right)+1+(x+x^2+....+x^n)\geq 1+2n$$

$$\Rightarrow \frac{(1+x+...+x^{n-1}+x^n)+x^{n+1}+x^{n+2}+.....+x^{2n}}{x^n}\geq 1+2n$$

$$\Rightarrow \quad \frac{x^n}{1+x+....+x^{2n}} \le \frac{1}{1+2n}.$$

17. (d) : $\displaystyle\int_2^3 \frac{xdx}{\sqrt{1+\left(x-\frac{1}{x}\right)^2}} - \int_{1/2}^{1/3} \frac{dx}{\sqrt{1+\left(x-\frac{1}{x}\right)^2}}$

$$= \int_2^3 \frac{xdx}{\sqrt{1+\left(x-\frac{1}{x}\right)^2}} + \int_2^3 \frac{t^3 dt}{t^2\sqrt{1+\left(\frac{1}{t}-t\right)^2}}$$

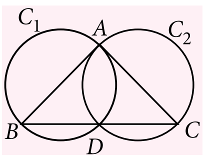
$$= \int_2^3 \frac{xdx}{\sqrt{1+\left(x-\frac{1}{x}\right)^2}} + \int_2^3 \frac{tdt}{\sqrt{1+\left(t-\frac{1}{t}\right)^2}} = I + I = 2I,$$

As $I = \displaystyle\int_2^3 \frac{x^2dx}{\sqrt{x^4-x^2+1}} = \int_2^3 \frac{xdx}{\sqrt{x^2-1+\frac{1}{x^2}}}$

$$= \int_2^3 \frac{xdx}{\sqrt{1+\left(x-\frac{1}{x}\right)^2}} = \int_2^3 \frac{tdt}{\sqrt{1+\left(t-\frac{1}{t}\right)^2}}.$$

18. (d) : $2y\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2 + y^2 = 0.$

19. (d) : Let their point of intersection other than A be D. As AB is a diameter of C_1



$\therefore \quad \angle ACD = \dfrac{\pi}{2}$

AC is a diameter of C_2

$\therefore \quad \angle ADC = \dfrac{\pi}{2}$

So, $\angle BDC = \pi$

$\Rightarrow$ B, D, C are collinear.

Hence, it's a certain case. $\therefore$ Probability = 1.

20. (c): $D > 0 \Rightarrow (a-3)^2 + 4(a+2) > 0$

$\Rightarrow a^2 - 6a + 9 + 4a + 8 > 0$

$\Rightarrow a^2 - 2a + 17 > 0 \quad \Rightarrow a \in R$

So, $\dfrac{a^2+1}{a^2+2} = 1 - \dfrac{1}{a^2+2} \ge \dfrac{1}{2}.$

21. (b) : $\vec{z} + \vec{z} \times \vec{x} = \vec{y} \quad \Rightarrow \quad |\vec{z} + \vec{z} \times \vec{x}|^2 = |\vec{y}|^2$

$\Rightarrow \quad (\vec{z} + \vec{z} \times \vec{x}) \cdot (\vec{z} + \vec{z} \times \vec{x}) = |\vec{y}|^2 = 1$

$\Rightarrow \quad |z^2| + |z|^2 |x|^2 \sin^2\theta = 1$

$\Rightarrow |z| = \dfrac{1}{\sqrt{1+\sin^2\theta}} = \dfrac{2}{\sqrt{7}} \Rightarrow \sin\theta = \dfrac{\sqrt{3}}{2} \Rightarrow \theta = \dfrac{\pi}{3} = 60°.$

22. (a) : $h'(x) = \dfrac{m}{n} \cdot x^{\frac{m-n}{n}} = \dfrac{m}{n} = \dfrac{m}{n} \cdot x^{-\left(\frac{even}{odd}\right)}$

So, $h(x)$ is undefined at $x = 0$ and $h'(x)$ does not change its sign in the neighbourhood. So, no extremums.

23. (b) : Image of A about $y = x$, $y = -2x$ are A_1 and A_2 which lies on BC.

$$A_1 \equiv (2,\ 1),\ A_2 \equiv \left(-\frac{11}{5},\ \frac{2}{5}\right)$$

Equation of BC is $x - 7y + 5 = 0$

$$r = \left|\frac{5}{\sqrt{1+49}}\right| = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

24. (b) : $\displaystyle\lim_{x\to 0} \frac{\tan(\pi\sec^2 x)}{x\sin^{-1}x} = \lim_{x\to 0} \frac{-\tan(\pi - \pi\sec^2 x)}{x\sin^{-1}x}$

$$= \lim_{x\to 0} \frac{\tan(\pi\tan^2 x)}{x\sin^{-1}x}$$

$$= \lim_{x\to 0} \frac{\tan(\pi\tan^2 x)}{\pi\tan^2 x} \times \frac{\pi\tan^2 x}{x^2} \times \frac{x^2}{x\sin^{-1}x} = \pi.$$

25. (c): Number of ways to choose A and B

$= 2^n \cdot 2^n = 2^{2n}.$

The number of subsets which contain exactly r elements in ${}^nC_r$.

$\therefore$ Number of ways to choose A and B such that they have same number of elements is

$$({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \ldots + ({}^nC_n)^2 = {}^{2n}C_n$$

$\therefore$ Required probability $= \dfrac{{}^{2n}C_n}{2^{2n}} = \dfrac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2^n \cdot n!}$

PART B

1. Let $\sin\dfrac{A}{2} + \sin\dfrac{B}{2} + \sin\dfrac{C}{2} = k$

or, $\quad 2\sin\dfrac{A+B}{4}\cos\dfrac{A-B}{4} + \cos\dfrac{A+B}{4} = k$

$\Rightarrow \quad 2\sin^2\dfrac{A+B}{4} - 2\cos\dfrac{A-B}{4}\sin\dfrac{A+B}{4} + k - 1 = 0$

Since, $\sin\dfrac{A+B}{4}$ is real, $-4\cos^2\dfrac{A-B}{4} - 8(k-1) \ge 0$

$\Rightarrow \quad 2(k-1) \le \cos^2\dfrac{A-B}{4} \le 1 \Rightarrow k \le \dfrac{3}{2}$

Hence, $2\cos\dfrac{A-B}{4}\sin\dfrac{A+B}{4} - 2\sin^2\dfrac{A+B}{4} + 1 \le \dfrac{3}{2}$

or, $\quad 2\sin\dfrac{A+B}{4}\left[\cos\dfrac{A-B}{4} - \sin\dfrac{\pi - C}{4}\right] \le \dfrac{1}{2}$

$$\Rightarrow\ 2\sin\frac{A+B}{4}\left[\cos\frac{A-B}{4}-\cos\frac{\pi+C}{4}\right]\le\frac{1}{2}$$

$$\text{or, } 4\sin\frac{A+B}{4}\sin\frac{\pi+C+A-B}{8}\sin\frac{\pi+C-A+B}{8}\le\frac{1}{2}$$

$$\text{or, } 4\sin\frac{\pi-C}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-A}{4}\le\frac{1}{2}$$

$$\Rightarrow\ \cos\frac{\pi+C}{4}\cos\frac{\pi+B}{4}\cos\frac{\pi+A}{4}\le\frac{1}{8}.$$

2. Clearly, θ_1, θ_0 are the roots of $\frac{\cos\theta}{\cos\theta_2}+\frac{\sin\theta}{\sin\theta_2}=1$

$$\frac{\cos\theta}{\cos\theta_2}=1-\frac{\sin\theta}{\sin\theta_2}$$

$$\Rightarrow\ \frac{\cos^2\theta}{\cos^2\theta_2}=1+\frac{\sin^2\theta}{\sin^2\theta_2}+\left(1-\frac{1}{\cos^2\theta_2}\right)=0$$

As the roots of this equation are θ_0 and θ_1

$$\Rightarrow\ \sin\theta_0\cdot\sin\theta_1=\frac{(\cos^2\theta_2-1)\cdot\cos^2\theta_2\cdot\sin^2\theta_2}{\cos^2\theta_2(\sin^2\theta_2+\cos^2\theta_2)}=-\sin^4\theta_2$$

$$\Rightarrow\ \frac{\sin\theta_0\cdot\sin\theta_1}{\sin^2\theta_2}=-\sin^2\theta_2$$

Similarly making a quadratic in $\cos\theta$, we get

$$\left(\frac{\cos\theta_0\cdot\cos\theta_1}{\cos^2\theta_2}\right)=-\cos^2\theta_2$$

$$\Rightarrow\ \frac{\cos\theta_0\cdot\cos\theta_1}{\cos^2\theta_2}+\frac{\sin\theta_0\cdot\sin\theta_1}{\sin^2\theta_2}=-1.$$

3. Let $P\equiv(h, k)$ be the foot of the perpendicular

Now, $lh + mk = 1$

and $m_{OP}\cdot m_{AB}=-1\ \Rightarrow\ \frac{k}{h}\cdot\left(-\frac{i}{m}\right)=-1$

$\Rightarrow\ kl = mh$

We now get, $m=\frac{k}{h^2+k^2}$; $l=\frac{h}{h^2+k^2}$

Making AB homogeneous with curve, we get equation of OA and OB as

$$x^2y^2 + (2gx + 2fy)(lx + my) + c(lx + my)^2 = 0$$

$$\Rightarrow\ C_{x^2}+C_{y^2}=0 \qquad [\because \angle AOB = 90°]$$

$$\Rightarrow\ l - 2gl + cl^2 + l + 2fm + cm^2 = 0$$

$$\Rightarrow\ 2+2g\frac{h}{h^2+k^2}+2f\frac{k}{h^2+k^2}+c\frac{h^2+k^2}{(h^2+k^2)^2}=0$$

$\Rightarrow$ Locus of P is $2x^2 + 2y^2 + 2gx + 2fy + c = 0.$

4. Let the centre be origin complex no. associated with A_1, B_1, C_1 and A_2, B_2, C_2 are respectively z_1, z_2, z_3 and z_4, z_5, z_6.

Complex no. associated with p_1 is z_7 and p_2 is z_8.

Now, z_1, z_2, z_3 and z_7 are concentric, hence

$$|z_1| = |z_2| = |z_3| = |z_7| = r_1 \text{ (radius of } \tau_1)$$

Similarly, $|z_4| = |z_5| = |z_7| = |z_8| = r_2$ (radius of τ_2)

Now, $(P_2A_1)^2 = |z_1-z_8|^2=(z_1-z_8)(\bar{z}_1-\bar{z}_8)$

$$= |z_1|^2+|z_8|^2-z_1\bar{z}_8-\bar{z}_1z_8$$

Similarly, $(P_2B_1)^2 = |z_2-z_8|^2=(z_2-z_8)(\bar{z}_2-\bar{z}_8)$

$$=|z_2|^2+|z_8|^2-z_2\bar{z}_8-\bar{z}_2z_8$$

$$(P_2C_1)^2=|z_3-z_8|^2=(z_3-z_8)(\bar{z}_3-\bar{z}_8)$$

$$=|z_3|^2+|z_8|^2-z_3\bar{z}_8-\bar{z}_3z_8$$

Adding all of them we get

$$(P_2A_1)^2 + (P_2B_1)^2 + (P_2C_1)^2$$

$$= |z_1|^2 + |z_2|^2 + |z_3|^2 + 3|z_8|^2 = 3(r_1^2 + r_2^2)$$

which is symmetric in r_1 and r_2. Hence the result holds.

5. Here, H is the orthocentre, m is mid-point of BC.

Let, D be the circumcentre.

Complex number associated with A, B, C are z_1, z_2, z_3 respectively.

$\Rightarrow\ |z_1| = |z_2| = |z_3| = R$ (Circumradius)

and $m=\frac{z_2+z_3}{2}$

Let the complex number associate with T be t

$$H \equiv z_1 + z_2 + z_3 \text{ (orthocentre)}$$

Now, $HM = MT$ (given)

$$\Rightarrow\ \frac{z_1+z_2+z_3+t}{2}=\frac{z_2+z_3}{2}\ \Rightarrow\ t = -4$$

$$\Rightarrow\ AT = |-z_1 - z_1| = |-2z_1| = 2R \text{ and } \frac{BC}{\sin 30°}=2R$$

$$\Rightarrow\ BC=2R=\frac{1}{2}=R\ \Rightarrow\ AT=2BC.$$

6. Let the tangent be $y=m_1x+\sqrt{a^2m_1^2-b^2}$

$$y=m_2x+\sqrt{a^2m_2^2-b^2}$$

Points of intersection of these tangents with axes are

$$\left(\frac{-\sqrt{a^2m_1^2-b^2}}{m_1}\right),\left(0, \sqrt{a^2m_1^2-b^2}\right),\left(\frac{-\sqrt{a^2m_2^2-b^2}}{m_2}, 0\right),$$

$$\left(0, \sqrt{a^2m_2^2-b^2}\right)$$

Now as four points are concyclic

$$\left(\frac{-\sqrt{a^2m_1^2-b^2}}{m_1}\right)\left(\frac{-\sqrt{a^2m_2^2-b^2}}{m_2}, 0\right)$$

$$= \sqrt{a^2m_1^2 - b^2}\sqrt{a^2m_2^2 - b^2}$$

$\Rightarrow$ $m_1m_2 = 1$.

7. Let the parabola be $y^2 = 4ax$.

ΔPQR is right angled at R.

Coordinates of $R = \left(-a, a\left(t - \frac{1}{t}\right)\right)$ and the coordinates of the centroid $(G) = \left(\frac{a}{3}\left(t_1^2 + \frac{1}{t_1^2} - 1\right), a\left(t - \frac{1}{t}\right)\right)$

Hence, the slope of line $RG = 0$.

8. The given expression is $(x^3 + 1)^2 + (x^3 + 1) + 4 = 0$

Discriminant of the above equation is less than zero *i.e.* $D < 0$.

Then, we have six complex roots and no real roots.

If $D \geq 0$, $x^3 + 1 = t$, then the equation reduces to $f(t) = t^2 + at + 4 = 0$

we will get two real roots and other roots will be complex except when $t = 1$ is one of the root

$\Rightarrow$ $f(1) = 0$ $\Rightarrow$ $a = -5$.

9. Consider, $g(x) = \sqrt{\frac{1+x^4}{1+5x^{10}}}$

Also $g(x)$ is positive $\forall$ $x \in R$ and $g(x)$ is continuous and $g(0) = 1$ and $\lim_{x\to\infty} g(x) = 0$.

$\Rightarrow$ $g(x)$ can take all values from $(0, 1]$

$\Rightarrow$ Range of $f(x) = \sin^{-1}(g(x))$ is $\left(0, \frac{\pi}{2}\right]$.

10. The angle subtended by the side BC at the orthocentre, the circumcentre and the incentre are $B + C$, $2A$ and $90° + \frac{A}{2}$ respectively.

If $\angle A = 60°$, then $B + C = 2A = 90° + \frac{A}{2} = 120°$

$\Rightarrow$ Angle subtended by BC at orthocentre, circumcentre and incentre are equal.

11. Equation of the plane through the lines $x + 2y - 4z = 0$ and $2x - y + 2z = 0$ is $x + 2y - 4z + \lambda(2x - y + 2z) = 0$ ... (i)

If $(1, 1, 1)$ lies on this plane, then $-1 + 3\lambda = 0$

$\Rightarrow$ $\lambda = 1/3$. So, the plane becomes

$3x + 6y - 12z + 2x - y + 2z = 0$

$\Rightarrow$ $x + y - 2z = 0$. ... (ii)

Also (i) will be perpendicular to (ii)

if $1 + 2\lambda + 2 - \lambda - 2(-4 + 2\lambda) = 0 \Rightarrow \lambda = \frac{11}{3}$

$\Rightarrow$ Equation of plane perpendicular to (ii) is

$5x - y + 2z = 0$. ...(iii)

Therefore the equation of line through $(1, 1, 1)$ and perpendicular to the given line is parallel to the normal to the plane (iii). Hence, the required line is

$$\frac{x-1}{5} = \frac{y-1}{-1} = \frac{z-1}{2}.$$

12. (i) $\cos^2x + \cos^2 2x + \cos^2 3x = 1$

$\Rightarrow$ $2\cos^2x + 2\cos^2 2x + 2\cos^2 3x = 2$

$\Rightarrow$ $\cos 2x + \cos 6x + 2\cos^2 2x = 0$

$\Rightarrow$ $2\cos 4x \cos 2x + 2\cos^2 2x = 0$

$\Rightarrow$ $\cos 2x(\cos 4x + \cos 2x) = 0$

$\Rightarrow$ $2\cos x \cos 2x \cos 3x = 0$

$\Rightarrow$ $x = (2m+1)\frac{\pi}{2}$ or $x = (2n+1)\frac{\pi}{4}$

or $x = (2k + 1)\frac{\pi}{6}$ where m, n and $k \in I$.

Alternative solution :

$\cos^2x + \cos^2 2x + \cos^2 3x = 1$

Let $z = \cos x + i\sin x$

$\Rightarrow$ $(z + z^{-1})^2 + (z^2 + z^{-2})^2 + (z^3 + z^{-3})^2 = 4$

$\Rightarrow$ $z^2 + z^{-2} + z^4 + z^{-4} + z^6 + z^{-6} + 2 = 0$

$\Rightarrow$ $z^{-6} + z^{-4} + z^{-2} + 1 + z^2 + z^4 + z^6 = -1$

$\Rightarrow$ $z^{-6}\frac{(1-(z^2)^7)}{1-z^2} = -1$

$\Rightarrow$ $z^{-6} - z^8 = -(1 - z^2) \Rightarrow z^7 - z^{-7} = -(z - z^{-1})$

$\Rightarrow$ $\sin 7\theta = -\sin\theta$ or $\sin 7\theta = \sin(-\theta)$

$\Rightarrow$ $7\theta = n\pi + (-1)^n(-\theta)$

Let $n = 2m$ $\Rightarrow 7\theta = 2m\pi - \theta$

$\Rightarrow$ $\theta = \frac{m\pi}{4}$, Here, $m \in I$ and $m \neq 4k$, $k \in I$

Let $n = (2m + 1)$

$\Rightarrow$ $7\theta = (2m + 1)\pi + \theta$ $\Rightarrow \theta = (2m+1)\frac{\pi}{6}$

(ii) It is evident from the inequality that

$\left|\sqrt{1+\sin 2x} - \sqrt{1-\sin 2x}\right| \leq \sqrt{2}$, $x \in [0, 2\pi]$

as $\left|\sqrt{1+\sin x} - \sqrt{1-\sin x}\right| \leq \sqrt{1+\sin x} \leq \sqrt{2}$

Now, $2\cos x \leq \left|\sqrt{1+\sin 2x} - \sqrt{1-\sin 2x}\right|$ holds for all x for which $\cos x \leq 0$.

$\Rightarrow$ $x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Now, if $\cos x > 0$, then

$4\cos^2 x \leq 1 + \sin 2x + 1 - \sin 2x - 2\sqrt{1-\sin^2 2x}$

$\Rightarrow$ $2 + 2\cos 2x \leq 2 - 2|\cos 2x|$

$\Rightarrow$ $|\cos 2x| \leq -\cos 2x \Rightarrow$ $x \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right] \cup \left[\frac{5\pi}{4}, \frac{7\pi}{4}\right]$

Hence, $x \in \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$

MATH archives

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE. In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. $\int \frac{x^2-1}{(x^2+1)\sqrt{x^4+1}}dx =$

(a) $\sec^{-1}\left(\frac{x^2+1}{\sqrt{2}x}\right)+c$ (b) $\frac{1}{\sqrt{2}}\sec^{-1}\left(\frac{x^2+1}{\sqrt{2}x}\right)+c$

(c) $\frac{1}{\sqrt{2}}\sec^{-1}\left(\frac{x^2+1}{\sqrt{2}}\right)+c$ (d) None of these

2. Let $I_1=\int_0^1 \frac{e^x}{1+x}dx$ and $I_2=\int_0^1 \frac{x^2}{e^{x^3}(2-x^3)}dx$, then $\frac{I_1}{I_2}=$

(a) $\frac{3}{e}$ (b) $\frac{e}{3}$ (c) $3e$ (d) $\frac{1}{3e}$

3. If $\sum_{i=1}^{18}(x-8)=9$ and $\sum_{i=1}^{18}(x_i-8)^2=45$ then the standard deviation of the observations $x_1, x_2, \ldots, x_{18}$ is

(a) 4/9 (b) 9/4 (c) 3/2 (d) 2/3

4. The distance of the point (1,2,3) from the plane $x+y+z=11$ measured parallel to the line $\frac{x+1}{1}=\frac{y-12}{-2}=\frac{z-7}{2}$ is

(a) 5 (b) 10 (c) 15 (d) 20

5. The diameter of the circle having the pair of lines $x^2+2xy+3x+6y=0$ as its normals and having the size just sufficient to contain the circle $x(x-4)+y(y-3)=0$ is

(a) 10 (b) 15 (c) 7 (d) 11

6. The point of intersection of two tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the product of whose slopes is c^2, lies on the curve,

(a) $y^2-b^2=c^2(b^2+a^2)$
(b) $y^2+b^2=c^2(b^2-a^2)$
(c) $x^2+b^2=c^2(b^2-a^2)$
(d) $y^2-a^2=c^2(b^2+a^2)$

7. If $f(x)$ is the solution of the equation $\frac{dy}{dx}=-2x(y-1)$ with $f(0)>1$, then $\underset{x\to\infty}{\mathrm{Lt}}\ f(x)$ is

(a) 0 (b) 2
(c) 1 (d) doesn't exist

8. Length of the normal chord of the parabola $y^2=4x$ which makes an angle of $\frac{\pi}{4}$, with the x-axis is

(a) 8 (b) $8\sqrt{2}$ (c) 4 (d) $4\sqrt{2}$

9. $\underset{x\to\infty}{\mathrm{Lt}}\ \frac{x^n}{e^x}=0$ (n is an integer) for

(a) no value of n
(b) all values of n
(c) only negative values of n
(d) only positive values of n

10. Statement-1: $f:A\to B$ and $g:B\to C$ are any functions then $(gof)^{-1}=f^{-1}og^{-1}$

Statement-2 : $f:A\to B$ and $g:B\to C$ are bijections then f^{-1}, g^{-1} are also bijections.

By : **Prof. Shyam Bhushan,** Director, Narayana IIT Academy, Jamshedpur. Mob. : 09334870021

(a) Statement-1, Statement-2 are correct, Statement-2 is correct explanation to Statement-1.
(b) Statement-1, Statement-2 are correct, Statement-2 is not correct explanation to Statement-1.
(c) Statement-1 is true, Statement-2 is false.
(d) Statement-1 is false, Statement-2 is true.

SOLUTIONS

1. (b) : $\int \frac{x^2-1}{x^2\left(x+\frac{1}{x}\right)\sqrt{x^2+\frac{1}{x^2}}}dx$

$=\int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)\sqrt{\left(x+\frac{1}{x}\right)^2-2}}dx$

Put $x+\frac{1}{x}=t$

$=\int \frac{dt}{t\sqrt{t^2-2}}$ [Put $t=\sqrt{2}\sec\theta$]

$=\int \frac{\sqrt{2}\sec\theta\tan\theta d\theta}{\sqrt{2}\sec\theta\sqrt{2}\tan\theta}=\frac{1}{\sqrt{2}}\theta+c$

$=\frac{1}{\sqrt{2}}\frac{\sec^{-1}\left(x+\frac{1}{x}\right)}{\sqrt{2}}+c=\frac{1}{\sqrt{2}}\sec^{-1}\left(\frac{x^2+1}{\sqrt{2}x}\right)+c$

2. (c) : Put $x^3 = t$

$\therefore\ I_2=\frac{1}{3}\int_0^1\frac{dt}{e^t(2-t)}=\frac{1}{3}\int_0^1\frac{e^t}{e(1+t)}dt=\frac{1}{3e}I_1 \Rightarrow \frac{I_1}{I_2}=3e$

3. (c) : Let $x_i - 8 = x$. Then, $\sum_{i=1}^{18}x=9,\ \sum_{i=1}^{18}x^2=45$

$\therefore\ \text{S.D}=\sqrt{\frac{\sum x_i^2}{n}-\left(\frac{\sum x_i}{n}\right)^2}=\sqrt{\frac{9}{4}}=\frac{3}{2}$

4. (c) : $\frac{x-1}{1}=\frac{y-2}{-2}=\frac{z-3}{2}=\lambda$

$\therefore$ Point on the plane $\big\}(\lambda+1,-2\lambda+2,2\lambda+3)$

$= \lambda + 6 = 11 \ \therefore\ \lambda = 5$

(6, –8, 13)

5. (b) : Normals $\Rightarrow \left(-3,\frac{3}{2}\right)$

Centre of given circle = (2, 3/2)

Radius = 5/2

Radius of required circle = 5 + 5/2 = 15/2

Diameter =15.

6. (b) : $y=mx\pm\sqrt{a^2m^2-b^2}$

$y-mx=\pm\sqrt{a^2m^2-b^2}=0$

$\Rightarrow y^2 + m^2x^2 - 2mxy = a^2m^2 - b^2$

$\Rightarrow k^2 + m^2b^2 - 2mhk = a^2m^2 - b^2$

$\Rightarrow m^2(b^2 - a^2) - 2mhk + b^2 + k^2 = 0$

$\Rightarrow \frac{b^2+k^2}{b^2-a^2}=c^2 \quad \therefore\ b^2 + y^2 = c^2(b^2 - a^2)$

7. (c) : $\frac{dy}{dx}+2xy=2x$

$e^{\int 2xdx}=e^{x^2}$

$ye^{x^2}=\int e^{x^2}2xdx=e^{x^2}+c \Rightarrow y=1+\frac{c}{e^{x^2}}$

$\underset{x\to\infty}{\text{Lt}}\ y=1$

8. (a)

9. (b) : Successive application of L 'hospital's rule.

10. (d) : Function need not be invertible.

MATHS MUSING

SOLUTION SET-172

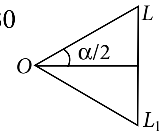
1. **(c) :** The lines are

$$\frac{x-b}{a}=\frac{y}{1}=\frac{z-d}{c} \;\&\; \frac{x-b'}{a'}=\frac{y}{1}=\frac{z-d'}{c'}$$

The d.r's are a, 1, c and a', 1, c'.
The lines are perpendicular, so, $aa' + 1 + cc' = 0$

2. **(b) :** $2ae = 8$ ....(i), $\quad \frac{2a}{e} = 10 \quad$ ...(ii)

Multiplying eqn. (i) & (ii), we get $4a^2 = 80$



$\therefore \quad a = 2\sqrt{5}, ae = 4 \;\Rightarrow\; e = \frac{2}{\sqrt{5}}$

$$\tan\frac{\alpha}{2} = \frac{a(1-e^2)}{ae} = \frac{1-e^2}{e} = \frac{1}{2\sqrt{5}}$$

$$\therefore \quad \tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}} = \frac{\frac{1}{\sqrt{5}}}{1-\frac{1}{20}} = \frac{4\sqrt{5}}{19}$$

$$\Rightarrow \quad \cos\alpha = \frac{19}{21} \Rightarrow \alpha = \cos^{-1}\left(\frac{19}{21}\right).$$

3. **(a) :** Let $f(x) = \frac{a}{3}x^3 + \frac{b}{2}x^2 + cx$

$$\Rightarrow f(0) = 0, f(1) = \frac{a}{3}+\frac{b}{2}+c = \frac{1}{6}[2a+3b+6c] = 0$$

$\therefore$ By Rolle's theorem, $f'(x) = 0$ has a root in (0, 1)
$\Rightarrow$ $ax^2 + bx + c = 0$ has a root in (0, 1).

4. **(d) :** The numbers to be selected are from the set {6, 12, 18, …, 96} of 16 numbers.

$$\therefore \quad \text{Required probability} = \frac{16C_3}{100C_3} = \frac{16\cdot15\cdot14}{100\cdot99\cdot98} = \frac{4}{1155}.$$

5. **(d) :** Given equation is homogeneous differential equation. So, put $y = vx$

$$\therefore \quad \frac{xdv}{dx} = v^3 + v^2\sqrt{v^2-1} - v$$

$$\Rightarrow \quad \frac{dx}{x} = \left(\frac{1}{\sqrt{v^2-1}} - \frac{1}{v}\right)dv$$

Integrating on both sides, we get

$$cx = \frac{v+\sqrt{v^2-1}}{v} = \frac{y+\sqrt{y^2-x^2}}{y}$$

Now, $y = 1$ when $x = 1 \;\Rightarrow\; c = 1$

$$\therefore \quad y = 2 \Rightarrow 2(x-1) = \sqrt{4-x^2} \;\Rightarrow x = \frac{8}{5}.$$

6. **(d) :** Let $t = 2^{\sin^2 x}$, then given equation becomes

$$at + \frac{a}{t} - 2 = 0 \Rightarrow at^2 - 2t + a = 0$$

On solving, $t = \frac{1 \pm \sqrt{1-a^2}}{a}$, $t = 2^{\sin^2 x} \in [1, 2]$

$$\therefore \quad 1 \le \frac{1 \pm \sqrt{1-a^2}}{a} \le 2 \Rightarrow a \le 1$$

and $\pm\sqrt{1-a^2} \le 2a - 1 \Rightarrow 5a^2 \ge 4a \Rightarrow a \ge \frac{4}{5}$

$$\therefore \quad a \in \left[\frac{4}{5}, 1\right].$$

7. **(d) :** $A_1 = \frac{\pi - A}{2}$, $A_2 = \frac{\pi - A_1}{2} = \frac{\pi + A}{4}$,

$$A_3 = \frac{\pi - A_2}{2} = \frac{3\pi - A}{8}, A_4 = \frac{\pi - A_3}{2} = \frac{5\pi + A}{16}.$$

8. **(c) :** $\frac{B_1C_1}{BC} = \frac{2r\sin A_1}{2R\sin A} = \frac{4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}}{\sin A}$

$$= 2\sin\frac{B}{2}\sin\frac{C}{2} = \cos\left(\frac{B-C}{2}\right) - \sin\frac{A}{2}.$$

9. **(6) :** a, ar, ar^2 are in G.P. and $4a$, $5ar$, $4ar^2$ are in A.P.

$\therefore \quad 4(1 + r^2) = 10r \Rightarrow 2r^2 - 5r + 2 = 0 \Rightarrow r = 2, \frac{1}{2}$

$\because \quad a(1 + r + r^2) = 42$. At $r = 2$, $a = 6$ and at $r = 1/2$, $a = 24$

10. **(a) :** (P) $\quad y_1 = -\sin\left(\frac{1}{3}\sin^{-1}x\right)\frac{1}{3\sqrt{1-x^2}}$

$\Rightarrow \quad 9(1 - x^2)\,{y_1}^2 = 1 - y^2.$

Differentiating and dropping the factor $2y_1$, we get

$$(1-x^2)y_2 - xy_1 + \frac{1}{9}y = 0 \Rightarrow \lambda = \frac{1}{9}.$$

(Q) $y(2) = 1$, $y'(2) = 0 \Rightarrow a = 1, b = 0$

$\quad y'(x) = 0 \Rightarrow x = -2, 2 \quad \therefore \quad y_{max} = y(-2) = \frac{1}{9}.$

(R) $y = ax^3 + bx^2 + cx + 5 \qquad \because \quad f(2) = f(-2) = 0$

$\Rightarrow \; -8a + 4b - 2c + 5 = 0$ ...(i) and $1 - 2a - 4b + c$ ...(ii)

Also, $\left[\frac{dy}{dx}\right]_{x=0} = 3 \quad \Rightarrow \quad c = 3 \qquad$ ...(iii)

On solving, we get $a = -\frac{1}{2}, b = -\frac{3}{4}, c = 3$

$$\Rightarrow \quad a+b+c = -\frac{1}{2} - \frac{3}{4} + 3 = \frac{7}{4}.$$

(S) Tangent at (x, y) is $Y - y = (X - x)\frac{dy}{dx}$

At (0, 0) $x^3 - 7x^2 + 6x + 5 = 3x^3 - 14x^2 + 6x$

$\Rightarrow 2x^3 - 7x^2 - 5 = 0.$

Let its roots be $x_1, x_2, x_3 \Rightarrow x_1 \cdot x_2 \cdot x_3 = \frac{5}{2}.$

◈◈

JEE MAIN

SOLVED PAPER 2017

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1. The function $f : R \to \left[-\frac{1}{2}, \frac{1}{2}\right]$ defined as $f(x) = \frac{x}{1+x^2}$, is

(a) injective but not surjective
(b) surjective but not injective
(c) neither injective nor surjective
(d) invertible

2. If for a positive integer n, the quadratic equation $x(x+1) + (x+1)(x+2) + \ldots\ldots + (x+\overline{n-1})(x+n) = 10n$ has two consecutive integral solutions, then n is equal to

(a) 9 (b) 10 (c) 11 (d) 12

3. Let ω be a complex number such that $2\omega + 1 = z$ where $z = \sqrt{-3}$. If $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2 - 1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$, then k is equal to

(a) z (b) -1 (c) 1 (d) $-z$

4. If $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$, then adj $(3A^2 + 12A)$ is equal to

(a) $\begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$ (b) $\begin{bmatrix} 51 & 84 \\ 63 & 72 \end{bmatrix}$

(c) $\begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$ (d) $\begin{bmatrix} 72 & -84 \\ -63 & 51 \end{bmatrix}$

5. If S is the set of distinct values of 'b' for which the following system of linear equations

$$x + y + z = 1,\ x + ay + z = 1,\ ax + by + z = 0$$

has no solution then S is

(a) an infinite set
(b) a finite set containing two or more elements
(c) a singleton
(d) an empty set

6. A man X has 7 friends, 4 of them are ladies and 3 are men. His wife Y also has 7 friends, 3 of them are ladies and 4 are men. Assume X and Y have no common friends. Then the total number of ways in which X and Y together can throw a party inviting 3 ladies and 3 men, so that 3 friends of each of X and Y are in this party, is

(a) 468 (b) 469 (c) 484 (d) 485

7. The value of

$$({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + ({}^{21}C_4 - {}^{10}C_4) + \ldots\ldots\ldots + ({}^{21}C_{10} - {}^{10}C_{10})$$ is

(a) $2^{21} - 2^{10}$ (b) $2^{20} - 2^9$
(c) $2^{20} - 2^{10}$ (d) $2^{21} - 2^{11}$

8. For any three positive real numbers a, b and c, $9(25a^2 + b^2) + 25(c^2 - 3ac) = 15b(3a + c)$. Then

(a) b, c and a are in A.P.
(b) a, b and c are in A.P.
(c) a, b and c are in G.P.
(d) b, c and a are in G.P.

9. Let $a, b, c \in R$. If $f(x) = ax^2 + bx + c$ is such that $a + b + c = 3$ and $f(x + y) = f(x) + f(y) + xy$, $\forall\ x, y \in R$, then $\sum_{n=1}^{10} f(n)$ is equal to

(a) 165 (b) 190
(c) 255 (d) 330

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10. $\lim_{x\to\frac{\pi}{2}} \frac{\cot x - \cos x}{(\pi - 2x)^3}$ equals

(a) $\frac{1}{16}$ (b) $\frac{1}{8}$ (c) $\frac{1}{4}$ (d) $\frac{1}{24}$

11. If for $x \in \left(0, \frac{1}{4}\right)$, the derivative of $\tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ is $\sqrt{x}\cdot g(x)$, then $g(x)$ equals

(a) $\frac{3x\sqrt{x}}{1-9x^3}$ (b) $\frac{3x}{1-9x^3}$

(c) $\frac{3}{1+9x^3}$ (d) $\frac{9}{1+9x^3}$

12. The normal to the curve $y(x - 2)(x - 3) = x + 6$ at the point where the curve intersects the y-axis passes through the point

(a) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, -\frac{1}{3}\right)$

(c) $\left(\frac{1}{2}, \frac{1}{3}\right)$ (d) $\left(-\frac{1}{2}, -\frac{1}{2}\right)$

13. Twenty meters of wire is available for fencing off a flower-bed in the form of a circular sector. Then the maximum area (in sq. m) of the flower-bed, is

(a) 10 (b) 25 (c) 30 (d) 12.5

14. Let $I_n = \int \tan^n x\, dx, (n > 1)$. If $I_4 + I_6 = a \tan^5 x + bx^5 + C$, where C is a constant of integration, then the ordered pair (a, b) is equal to

(a) $\left(\frac{1}{5}, 0\right)$ (b) $\left(\frac{1}{5}, -1\right)$

(c) $\left(-\frac{1}{5}, 0\right)$ (d) $\left(-\frac{1}{5}, 1\right)$

15. The integral $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1+\cos x}$ is equal to

(a) 2 (b) 4 (c) –1 (d) –2

16. The area (in sq. units) of the region $\{(x, y) : x \geq 0, x + y \leq 3, x^2 \leq 4y \text{ and } y \leq 1 + \sqrt{x}\}$ is

(a) $\frac{3}{2}$ (b) $\frac{7}{3}$ (c) $\frac{5}{2}$ (d) $\frac{59}{12}$

17. If $(2+\sin x)\frac{dy}{dx} + (y+1)\cos x = 0$ and $y(0) = 1$, then $y\left(\frac{\pi}{2}\right)$ is equal to

(a) $-\frac{2}{3}$ (b) $-\frac{1}{3}$ (c) $\frac{4}{3}$ (d) $\frac{1}{3}$

18. Let k be an integer such that triangle with vertices $(k, -3k)$, $(5, k)$ and $(-k, 2)$ has area 28 sq. units. Then the orthocentre of this triangle is at the point

(a) $\left(1, \frac{3}{4}\right)$ (b) $\left(1, -\frac{3}{4}\right)$ (c) $\left(2, \frac{1}{2}\right)$ (d) $\left(2, -\frac{1}{2}\right)$

19. The radius of a circle, having minimum area, which touches the curve $y = 4 - x^2$ and the lines, $y = |x|$ is

(a) $2(\sqrt{2} - 1)$ (b) $4(\sqrt{2} - 1)$

(c) $4(\sqrt{2} + 1)$ (d) $2(\sqrt{2} + 1)$

20. The eccentricity of an ellipse whose centre is at the origin is $\frac{1}{2}$. If one of its directrices is $x = -4$, then the equation of the normal to it at $\left(1, \frac{3}{2}\right)$ is

(a) $4x - 2y = 1$ (b) $4x + 2y = 7$

(c) $x + 2y = 4$ (d) $2y - x = 2$

21. A hyperbola passes through the point $P(\sqrt{2}, \sqrt{3})$ and has foci at $(\pm 2, 0)$. Then the tangent to this hyperbola at P also passes through the point

(a) $(2\sqrt{2}, 3\sqrt{3})$ (b) $(\sqrt{3}, \sqrt{2})$

(c) $(-\sqrt{2}, -\sqrt{3})$ (d) $(3\sqrt{2}, 2\sqrt{3})$

22. The distance of the point $(1, 3, -7)$ from the plane passing through the point $(1, -1, -1)$, having normal perpendicular to both the lines $\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-4}{3}$ and $\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z+7}{-1}$ is

(a) $\frac{10}{\sqrt{83}}$ (b) $\frac{5}{\sqrt{83}}$ (c) $\frac{10}{\sqrt{74}}$ (d) $\frac{20}{\sqrt{74}}$

23. If the image of the point $P(1, -2, 3)$ in the plane, $2x + 3y - 4z + 22 = 0$ measured parallel to the line; $\frac{x}{1} = \frac{y}{4} = \frac{z}{5}$ is Q, then PQ is equal to

(a) $2\sqrt{42}$ (b) $\sqrt{42}$ (c) $6\sqrt{5}$ (d) $3\sqrt{5}$

24. Let $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$. Let $\vec{c}$ be a vector such that $|\vec{c} - \vec{a}| = 3$, $|(\vec{a} \times \vec{b}) \times \vec{c}| = 3$ and the angle between $\vec{c}$ and $\vec{a} \times \vec{b}$ be 30°. Then $\vec{a} \cdot \vec{c}$ is equal to

(a) 2 (b) 5 (c) $\frac{1}{8}$ (d) $\frac{25}{8}$

25. A box contains 15 green and 10 yellow balls. If 10 balls are randomly drawn, one-by-one, with replacement, then the variance of the number of green balls drawn is

(a) 6 (b) 4 (c) $\frac{6}{25}$ (d) $\frac{12}{5}$

26. For three events A, B and C,
P(Exactly one of A or B occurs)
$= P$(Exactly one of B or C occurs)
$= P$(Exactly one of C or A occurs) $= \frac{1}{4}$
and P(All the three events occur simultaneously)
$= \frac{1}{16}$. Then the probability that at least one of the events occurs, is

(a) $\frac{7}{16}$ (b) $\frac{7}{64}$ (c) $\frac{3}{16}$ (d) $\frac{7}{32}$

27. If two different numbers are taken from the set {0, 1, 2, 3, ..., 10}; then the probability that their sum as well as absolute difference are both multiples of 4, is

(a) $\frac{12}{55}$ (b) $\frac{14}{45}$ (c) $\frac{7}{55}$ (d) $\frac{6}{55}$

28. If $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$, then the value of $\cos 4x$ is

(a) $\frac{1}{3}$ (b) $\frac{2}{9}$ (c) $-\frac{7}{9}$ (d) $-\frac{3}{5}$

29. Let a vertical tower AB have its end A on the level ground. Let C be the mid-point of AB and P be a point on the ground such that $AP = 2AB$. If $\angle BPC = \beta$, then $\tan\beta$ is equal to

(a) $\frac{1}{4}$ (b) $\frac{2}{9}$ (c) $\frac{4}{9}$ (d) $\frac{6}{7}$

30. The following statement $(p \to q) \to [(\sim p \to q) \to q]$ is
(a) equivalent to $\sim p \to q$ (b) equivalent to $p \to \sim q$
(c) a fallacy (d) a tautology

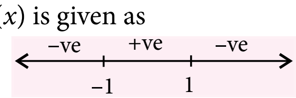
SOLUTIONS

1. (b) : We have $f(x) = \dfrac{x}{1+x^2}$

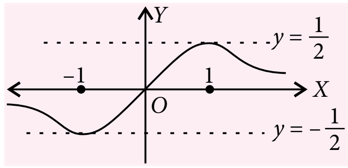
$$\therefore \quad f'(x) = \frac{(1+x^2)\cdot 1 - x\cdot 2x}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2} = \frac{(1-x)(1+x)}{(1+x^2)^2}$$

The sign of $f'(x)$ is given as



Now f can be graphed as under



Clearly function is surjective but not injective, as a horizontal line meet the curve in two points.

[Rating : Challenging]

2. (c) : We have, $\displaystyle\sum_{k=1}^{n} (x+k-1)(x+k) = 10n$

$$\Rightarrow \sum_{k=1}^{n} x^2 + (2k-1)x + k(k-1) = 10n$$

$$\Rightarrow nx^2 + n^2x + \frac{1}{3}n(n^2-1) = 10n$$

$$\Rightarrow x^2 + nx + \frac{1}{3}(n^2-1) - 10 = 0$$

$\Rightarrow 3x^2 + 3nx + n^2 - 31 = 0$

Let consecutive roots be n and $n + 1$, then

$(n + 1 - n)^2 = (n + 1 + n)^2 - 4n(n + 1)$

$$\Rightarrow 1 = n^2 - 4\left(\frac{n^2-31}{3}\right) \Rightarrow n^2 = 121 \;\therefore\; n = 11$$

[Rating : Challenging]

3. (d) : We have, $z = 1 + 2\omega$

i.e., $i\sqrt{3} = 1 + 2\omega \;\therefore\; \omega = \dfrac{-1 + i\sqrt{3}}{2}$

Then ω is a cube root of unity.

Also, $1 + \omega + \omega^2 = 0$

$$\text{Now} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3k \Rightarrow \begin{vmatrix} 3 & 1 & 1 \\ 0 & \omega & \omega^2 \\ 0 & \omega^2 & \omega \end{vmatrix} = 3k$$

$\Rightarrow 3(\omega^2 - \omega^4) = 3k$

$\Rightarrow k = \omega^2 - \omega = -1 - \omega - \omega = -1 - 2\omega = -z$

[Rating : Easy]

4. (a) : Given, $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$

Then, A satisfies the characteristic equation

$A^2 - 3A - 10I = 0$

Now $3A^2 + 12A = 3(3A + 10I) + 12A = 21A + 30I$

$$= \begin{bmatrix} 42 & -63 \\ -84 & 21 \end{bmatrix} + \begin{bmatrix} 30 & 0 \\ 0 & 30 \end{bmatrix} = \begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$$

$$\therefore \quad \text{adj}(3A^2+12A)=\begin{bmatrix}51 & 63\\ 84 & 72\end{bmatrix}$$

[Rating : Easy]

5. (c) : The equation can be written as

$$\begin{bmatrix}1 & 1 & 1\\ 1 & a & 1\\ a & b & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

$$\text{Let } \Delta=\begin{vmatrix}1 & 1 & 1\\ 1 & a & 1\\ a & b & 1\end{vmatrix}=\begin{vmatrix}1 & 1 & 1\\ 0 & a-1 & 0\\ a & b & 1\end{vmatrix}=-(1-a)^2$$

The necessary condition is $\Delta = 0 \Rightarrow a = 1$

But for $a = 1$ the equation becomes

$x + y + z = 1$

$x + y + z = 1$

$x + by + z = 0$

For no solution $b = 1$. Then S is a singleton set.

[Rating : Difficult]

6. (d) : We can do casework on number of ladies and men to be invited.

X, Y can satisfy the condition in 4 ways

(i) X invites 3 ladies and Y invites 3 men.

(ii) X invites 2 ladies, 1 man and Y invites 1 lady 2 men.

(iii) X invites 1 lady, 2 men and Y invites 2 ladies, 1 man.

(iv) X invites 3 men and Y invites 3 ladies.

The number of ways

$= {}^4C_3 \cdot {}^4C_3 + {}^4C_2 \cdot {}^3C_1 \cdot {}^3C_1 \cdot {}^4C_2 + {}^4C_1 \cdot {}^3C_2 \cdot {}^3C_2 \cdot {}^4C_1 + {}^3C_3 \cdot {}^3C_3$

$= 16 + 324 + 144 + 1 = 485.$

[Rating : Medium]

7. (c) : $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_{10} - {}^{10}C_{10})$

$= ({}^{21}C_1 + {}^{21}C_2 + ... + {}^{21}C_{10}) - ({}^{10}C_1 + {}^{10}C_2 + ... + {}^{10}C_{10})$

$$=\frac{2^{21}}{2}-2^{10}=2^{20}-2^{10}$$

[Rating : Medium]

8. (a) : $9(25a^2 + b^2) + 25(c^2 - 3ac) = 15b(3a + c)$

$\Rightarrow (15a)^2 + (3b)^2 + (5c)^2 - (15a)(5c) - (15a)(3b) - (3b)(5c) = 0$

$$\Rightarrow \frac{1}{2}[(15a-3b)^2+(3b-5c)^2+(5c-15a)^2]=0$$

$\Rightarrow (15a - 3b)^2 = 0, (3b - 5c)^2 = 0, (5c - 15a)^2 = 0$

$\Rightarrow 15a = 3b = 5c$

$$\Rightarrow \frac{a}{1}=\frac{b}{5}=\frac{c}{3} \Rightarrow b, c, a \text{ are A.P.}$$

[Rating : Medium]

9. (d) : Given, $f(x) = ax^2 + bx + c$

and $f(x + y) = f(x) + f(y) + xy$

$\Rightarrow a(x + y)^2 + b(x + y) + c = ax^2 + bx + c + ay^2 + by + c + xy$

$\Rightarrow 2axy = c + xy$

i.e., $(2a - 1)xy - c = 0 \ \forall\ x, y \in R$

Then, $a=\frac{1}{2}, c=0$

Also, $a + b + c = 3 \quad \therefore \quad b = 5/2$

Now, $f(x)=\frac{1}{2}x^2+\frac{5}{2}x$

$$\sum_{n=1}^{10} f(x)=\frac{1}{2}\sum_{n=1}^{10} n^2+\frac{5}{2}\sum_{n=1}^{10} n$$

$$=\frac{1}{2}\frac{10\cdot 11\cdot 21}{6}+\frac{5}{2}\frac{10\cdot 11}{2}=\frac{10\cdot 11}{12}[21+15]=330$$

Alternative solution :

Let $x = m$, $y = 1$ in $f(x + y) = f(x) + f(y) + xy$ to obtain

$\therefore \quad f(m + 1) = f(m) + f(1) + m = f(m) + 3 + m$

$\Rightarrow \quad f(m + 1) - f(m) = 3 + m$

Changing m to $m-1$, we get

$f(m) - f(m - 1) = 3 + (m - 1)$ ... (i)

$f(2) - f(1) = 3 + 1$ ... (ii)

Adding (i) and (ii), we get $f(m + 1) - 3 = 3 + \frac{m(m+1)}{2}$

$$f(m+1)=3m+\frac{m(m+1)}{2}+3$$

$$\therefore \quad f(m)=3(m-1)+\frac{(m-1)m}{2}+3=3m+\frac{m^2-m}{2}$$

$$=\frac{m^2+5m}{2}=\frac{m^2}{2}+\frac{5}{2}m$$

from here calculation is same as in previous solution.

[Rating : Challenging]

10. (a) : We have $\lim\limits_{x\to\frac{\pi}{2}} \frac{\cot x-\cos x}{(\pi-2x)^3}$

$$\lim_{h\to 0}\frac{(-\tan h)-(-\sin h)}{(-2h)^3}=-\frac{1}{8}\lim_{h\to 0}\frac{\sin h-\tan h}{h^3}$$

$$=\frac{1}{8}\lim_{h\to 0}\frac{\tan h(1-\cos h)}{h^3}$$

$$=\frac{1}{8}\lim_{h\to 0}\left(\frac{\tan h}{h}\right)\left(\frac{2\sin^2\frac{h}{2}}{h^2}\right)=\frac{1}{8}\cdot 1\cdot 2\cdot\frac{1}{4}=\frac{1}{16}.$$

[Rating : Easy]

11. (d) : Let $u = \tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ $x \in \left(0, \frac{1}{4}\right)$

$= \tan^{-1}\left(\frac{2\cdot(3x^{3/2})}{1-(3x^{3/2})^2}\right) = 2\tan^{-1}(3x^{3/2})$

This holds as $3x^{3/2} \in (0, 3/8)$

Differentiating with respect to x, we obtain

$$\frac{du}{dx} = 2\cdot\frac{1}{1+9x^3}\cdot 3\cdot\frac{3}{2}\cdot x^{\frac{1}{2}} = \frac{9\sqrt{x}}{1+9x^3}$$

$$\Rightarrow \quad \sqrt{x}\cdot g(x) = \frac{9\sqrt{x}}{1+9x^3}$$

$$\Rightarrow \quad g(x) = \frac{9}{1+9x^3}.$$

[Rating : Easy]

12. (a) : We have, $y(x-2)(x-3) = x+6$

It meets the y-axis where $x = 0$, *i.e.* $y(6) = 6$ $\therefore$ $y = 1$

The point of intersection is (0,1).

$$\text{Now, } y = \frac{x+6}{x^2-5x+6} \qquad \text{...(i)}$$

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(x^2-5x+6)\cdot 1-(x+6)\cdot(2x-5)}{(x^2-5x+6)^2}$$

$$\text{Now, } \left.\frac{dy}{dx}\right|_{x=0} = \frac{6-(6)(-5)}{36} = 1$$

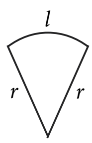
$\therefore$ Slope of normal = –1

Then the equation to curve is $y - 1 = -1(x - 0)$

i.e. $x + y - 1 = 0$.

[Rating : Easy]

13. (b) : Let r be the radius of circle and l the length of arc of the circle.



Now $l + 2r = 20$ (given)

Also $l = r\theta$ $\Rightarrow$ $\theta r + 2r = 20$

$$\therefore \theta = \frac{20-2r}{r}$$

$$\text{Now, } A = \frac{\pi r^2\theta}{2\pi} = \frac{r^2}{2}\cdot\frac{20-2r}{r} = r(10-r)$$

We have $\frac{dA}{dr} = 10 - 2r$

$$\frac{dA}{dr} = 0 \Rightarrow r = 5$$

Also, $\frac{d^2A}{dr^2} = -2 < 0$ $\therefore$ $A(r)$ is maximum at $r = 5$

Area = 5(10 – 5) = 25

Alternative solution :

We have, $A = r(10 - r)$

Applying A.M. & G.M. inequality, we get

$$\sqrt{r(10-r)} \le \frac{r+10-r}{2} \quad i.e., \sqrt{r(10-r)} \le 5$$

$\therefore$ $r(10 - r) \le 25$

Then the maximum area is 25 and is achieved at $r = 10 - r$ *i.e.* $r = 5$.

[Rating : Medium]

14. (a) : We have $I_n = \int \tan^n x\,dx, (n>1)$

$$= \int \tan^{n-2} x\,(\sec^2 x - 1)dx$$

$$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} dx$$

$$= \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

$$\text{Then, } I_n + I_{n-2} = \frac{(\tan x)^{n-1}}{n-1}$$

$$\text{Now, } I_4 + I_6 = \frac{\tan x^5}{5} \qquad \text{...(i)}$$

And $I_4 + I_6 = a \tan x^5 + bx^5 + C$ (Given) ...(ii)

On comparing (i) and (ii), we get

$a = \frac{1}{5}, b = 0,$ and C is a constant of integration.

[Rating : Easy]

15. (a) : Let, $I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1+\cos x}$

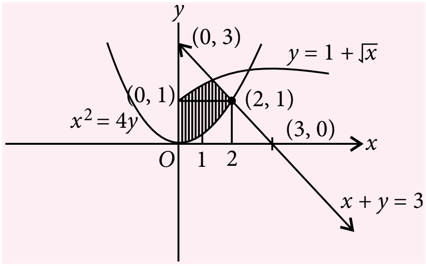
$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1+\cos(\pi-x)} = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1-\cos x}$$

On adding, we have, $2I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{2}{1-\cos^2 x}dx$

$$\Rightarrow \quad I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \operatorname{cosec}^2 x\,dx = -\cot x\Big|_{\pi/4}^{3\pi/4} = 2$$

[Rating : Medium]

16. (c) : The graph of the region is as follows :



Required area $= \int_0^1 (1+\sqrt{x})dx + \int_1^2 (3-x)dx - \int_0^2 \frac{x^2}{4}dx$

$$= x + \frac{2x^{3/2}}{3}\Bigg|_0^1 + 3x - \frac{x^2}{2}\Bigg|_1^2 - \frac{x^3}{12}\Bigg|_0^2$$

$$= \left(1+\frac{2}{3}\right) + \left(3\cdot 2 - \frac{2^2}{2} - 3\cdot 1 + \frac{1^2}{2}\right) - \frac{2^3}{12}$$

$$= \frac{5}{3} + \left(4 - \frac{5}{2}\right) - \frac{2}{3} = \frac{5}{2}.$$

[Rating : Challenging]

17. (d) : We have $\frac{dy}{dx} = -\frac{(y+1)\cos x}{2+\sin x}$

$$\int \frac{dy}{y+1} = -\int \frac{\cos x}{2+\sin x}dx$$

$\Rightarrow$ $\ln(y + 1) = -\ln(2 + \sin x) + \ln \lambda$

$\Rightarrow$ $(y + 1)(2 + \sin x) = \lambda$

As $y(0) = 1$ $\Rightarrow$ $2\cdot 2 = \lambda$ or $\lambda = 4$

At $x = \frac{\pi}{2}, y\left(\frac{\pi}{2}\right) = \frac{4}{2+1} - 1 = \frac{4}{3} - 1 = \frac{1}{3}.$

[Rating : Medium]

18. (c) : As area is given to be 56, we have

$$\begin{vmatrix} k & -3k & 1 \\ 5 & k & 1 \\ -k & 2 & 1 \end{vmatrix} = \pm 56$$

Expanding, we get

$k(k - 2) - 5(-3k - 2) - k(-3k - k) = \pm 56$

$\Rightarrow$ $k^2 - 2k + 15k + 10 + 3k^2 + k^2 = \pm 56$

$\Rightarrow$ $5k^2 + 13k + 10 = \pm 56$

Taking the positive sign $5k^2 + 13k - 46 = 0$

$\Rightarrow$ $(5k + 23)(k - 2) = 0$

$\therefore$ $k = 2$ is an integer

Taking the negative sign

$5k^2 + 13k + 66 = 0$

$D = 13^2 - 4\cdot 5\cdot 66 < 0$

Thus there is no solution in this case.

So the vertices are $A(2, -6)$, $B(5, 2)$ and $C(-2, 2)$.

The equation of altitude from A is $x = 2$ and

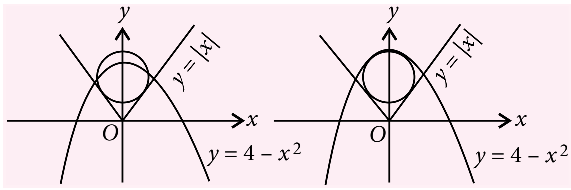
The equation of altitude from C is $y - 2 = -\frac{3}{8}(x+2)$

i.e., $3x + 8y - 10 = 0$

Solving the two we get the orthocentre as $\left(2, \frac{1}{2}\right)$.

[Rating : Challenging]

19. (b) : There are two possibilities



For both of them we get different answers.

For 1st case :

$x^2 + (y - b)^2 = r^2$ as $y = x$ is tangent to circle

$$\Rightarrow \left|\frac{0-b}{\sqrt{2}}\right| = r \quad \therefore \quad b = r\sqrt{2}$$

Now $x^2 + (y-b)^2 = \frac{b^2}{2}$

As $x^2 = 4 - y$

We have, $4 - y + (y-b)^2 = \frac{b^2}{2}$

Arranging as a quadratic in y, we have

$$y^2 - (2b+1)\,y + \frac{b^2}{2} + 4 = 0$$

The discriminant being zero yields

$$(2b+1)^2 - 4\left(\frac{b^2}{2} + 4\right) = 0$$

$\Rightarrow$ $4b^2 + 4b + 1 - 2b^2 - 16 = 0$

i.e. $2b^2 + 4b - 15 = 0$

$$\therefore \quad b = \frac{-4 \pm \sqrt{16+120}}{4} = \frac{-4 \pm 2\sqrt{34}}{4} = \frac{-2 \pm \sqrt{34}}{2}$$

Taking the positive value

$$b = \frac{\sqrt{34}-2}{2} \quad \therefore \; r = \frac{\sqrt{34}-2}{2\sqrt{2}}$$

For 2nd case

Co-ordinates of centre as $(0, 4-r)$, r being radius

$y = x$ touch the circle

$$\Rightarrow \left|\frac{0-(4-r)}{\sqrt{2}}\right| = r \quad \Rightarrow \quad r - 4 = \pm r\sqrt{2}$$

Which gives $r(1 + \sqrt{2}\,) = 4$ (As r can't be negative)

$$\therefore \quad r = \frac{4}{\sqrt{2}+1} = 4(\sqrt{2}-1)$$

Remark : The problem, as posed, is ambiguous because of choices. The best choice is (b). **[Rating : Medium]**

20. (a) : As $x = -\frac{a}{e} = -4$

We have, $a = 4e = 4 \cdot \frac{1}{2} = 2$

Again $b^2 = a^2(1-e^2) = a^2\left(1-\frac{1}{4}\right) = \frac{4 \cdot 3}{4} = 3$

Thus the equation to ellipse is $\frac{x^2}{4} + \frac{y^2}{3} = 1$

Differentiating w.r.t x, we get

$$\frac{2x}{4} + \frac{2y}{3} \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{3}{4}\frac{x}{y}$$

At $\left(1, \frac{3}{2}\right)$, $\frac{dy}{dx} = -\frac{3}{4} \cdot \frac{1 \cdot 2}{3} = -\frac{1}{2}$

So the slope of normal is 2. The equation is

$y - \frac{3}{2} = 2(x-1)$ *i.e.*, $4x - 2y - 1 = 0$ **[Rating : Easy]**

21. (a) : The equation to the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

We have, $ae = 2 \Rightarrow a^2e^2 = 4$

Also, $b^2 = a^2(e^2 - 1) \Rightarrow a^2 + b^2 = a^2e^2 = 4$

The hyperbola passes through $(\sqrt{2}, \sqrt{3})$ means

$$\frac{2}{a^2} - \frac{3}{b^2} = 1$$

On solving, we get $\frac{2}{a^2} - \frac{3}{4-a^2} = 1$

$\Rightarrow$ $2(4 - a^2) - 3a^2 = a^2(4 - a^2)$

$\Rightarrow$ $8 - 5a^2 = 4a^2 - a^4 = a^4 - 9a^2 + 8 = 0$

$\Rightarrow$ $(a^2 - 1)(a^2 - 8) = 0$ $\therefore$ $a^2 = 1$

As $a^2 = 8$ will give b^2 negative. $\therefore$ $a^2 = 1$ and $b^2 = 3$

So, equation of the hyperbola is $\frac{x^2}{1} - \frac{y^2}{3} = 1$

The equation of tangent at P is $\frac{x\sqrt{2}}{1} - \frac{y\sqrt{3}}{3} = 1$

The point $(2\sqrt{2}, 3\sqrt{3})$ lies on it.

[Rating : Challenging]

22. (a) : The normal vector to the plane is given by

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & -1 & -1 \end{vmatrix} = 5\hat{i} + 7\hat{j} + 3\hat{k}$$

The plane is given by $5(x - 1) + 7(y + 1) + 3(z + 1) = 0$

i.e., $5x + 7y + 3z + 5 = 0$

The distance of $(1, 3, -7)$ from the above plane is

$$\left|\frac{5 + 21 - 21 + 5}{\sqrt{5^2 + 7^2 + 3^2}}\right| = \frac{10}{\sqrt{83}}$$

[Rating : Easy]

23. (a) : The line PQ is given by

$$\frac{x-1}{1} = \frac{y+2}{4} = \frac{z-3}{5} = t$$

Let a point M on PQ be $(t + 1, 4t - 2, 5t + 3)$.

For this point to lie in the plane $2x + 3y - 4z + 22 = 0$ we have,

$2(t + 1) + 3(4t - 2) - 4(5t + 3) + 22 = 0$

$\Rightarrow$ $-6t + 6 = 0 \Rightarrow t = 1$

Then the point M is $(2, 2, 8)$

$\therefore$ $PQ = 2PM = 2\sqrt{1^2 + 4^2 + 5^2} = 2\sqrt{42}$

[Rating : Medium]

24. (a) : $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k} \Rightarrow |a| = 3$ and $\vec{b} = \hat{i} + \hat{j}$

Now, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\hat{i} - 2\hat{j} + \hat{k}$ $\therefore |\vec{a} \times \vec{b}| = 3$

We also have,

$$|(\vec{a} \times \vec{b}) \times \vec{c}| = |\vec{a} \times \vec{b}||\vec{c}||\sin 30°||\hat{n}| = 3|\vec{c}| \cdot \frac{1}{2}n$$

$\Rightarrow$ $3 = 3|\vec{c}| \cdot \frac{1}{2}$ $\therefore$ $|\vec{c}| = 2$

Since, $|\vec{c} - \vec{a}| = 3$ ...(i)

On squaring (i), we get $c^2 + a^2 - 2\vec{c} \cdot \vec{a} = 9$

$\Rightarrow$ $4 + 9 - 2\vec{a} \cdot \vec{c} = 9 \Rightarrow \vec{a} \cdot \vec{c} = 2$

[Rating : Challenging]

25. (d) : We know that variance $= npq$

p (probability of drawing a green ball) $= \frac{15}{25} = \frac{3}{5}$

Here, $n = 10$, $p = \frac{3}{5}, q = \frac{2}{5}$

Then, variance $= 10 \cdot \frac{3}{5} \cdot \frac{2}{5} = \frac{12}{5}$

[Rating : Easy]

26. (a) : Given P(exactly one of A or B occurs) $= \frac{1}{4}$

Then, $P(A) + P(B) - 2P(A \cap B) = \frac{1}{4}$

Similarly, $P(B) + P(C) - 2P(B \cap C) = \frac{1}{4}$

Also, $P(C) + P(A) - 2P(C \cap A) = \frac{1}{4}$

Adding all of them, we have

$P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) = \frac{3}{8}$

Now, $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

$= \frac{3}{8} + \frac{1}{16} = \frac{7}{16}$ **[Rating : Challenging]**

27. (d) : We have, $4/a - b$ and $4/a + b$

So the possibilities are

a	0	2	4	6	8	10
b	4, 8	6, 10	0, 8	2, 10	0, 4	2, 6

$\therefore$ Required probability $= \frac{6}{^{11}C_2} = \frac{6 \cdot 2}{11 \cdot 10} = \frac{6}{55}$

[Rating : Easy]

28. (c) : $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$

Let $u = \tan^2 x$, we have $5\left(u - \frac{1}{1+u}\right) = 2\left(\frac{1-u}{1+u}\right) + 9$

$\Rightarrow \quad 5(u^2 + u - 1) = 2 - 2u + 9 + 9u$

$\therefore \quad 5u^2 - 2u - 16 = 0 \quad \Rightarrow \quad (5u + 8)(u - 2) = 0$

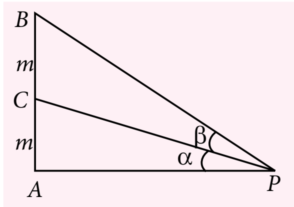
But u is positive $\therefore\ u = 2$

Now, $\tan^2 x = 2 \Rightarrow \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x} = \frac{1-2}{1+2} = \frac{-1}{3}$

$\Rightarrow \quad \cos 4x = 2\cos^2 2x - 1 = 2\left(\frac{1}{9}\right) - 1 = \frac{-7}{9}$

[Rating : Challenging]

29. (b) :



Let $\angle APC = \alpha$, we have

$\tan(\alpha + \beta) = \frac{AB}{AP} = \frac{1}{2}$

Now, $\tan\alpha = \frac{m}{4m} = \frac{1}{4}$

Now, $\tan\beta = \tan(\alpha + \beta - \alpha)$

$= \frac{\tan(\alpha+\beta) - \tan\alpha}{1 + \tan(\alpha+\beta)\tan\alpha} = \frac{\frac{1}{2} - \frac{1}{4}}{1 + \frac{1}{2}\cdot\frac{1}{4}} = \frac{\frac{1}{4}}{\frac{9}{8}} = \frac{2}{9}$

[Rating : Easy]

30. (d) : We have

$(p \to q) \to [(\sim p \to q) \to q]$ simplifying as

$(p \to q) \to ((p \vee q) \to q)$

$(p \to q)((\sim p \wedge \sim q) \vee q)$

$(p \to q) \to ((\sim p \vee q) \wedge (\sim q \vee q))$

$(p \to q) \to (p \to q)$

which is a tautology. **[Rating : Medium]**

CBSE BOARD SOLVED PAPER 2017

CLASS XII

Units	VSA(1 mark)	SA(2 marks)	VBQ(4 marks)	LA I(4 marks)	LA II(6 marks)	Total
Relations and Functions	---	---	---	---	6(1)	6(1)
Inverse Trigonometric Functions	---	---	---	4(1)	---	4(1)
Matrices	---	2(1)	---	---	---	2(1)
Determinants	1(1)	---	---	4(1) *	6(1)	11(3)
Continuity and Differentiability	1(1)	2(1)	---	4(1)	---	7(3)
Application of Derivatives	---	4(2)	---	---	6(1)	10(3)
Integrals	1(1)	2(1)	---	8(2)	---	11(4)
Application of Integrals	---	---	---	---	6(1)	6(1)
Differential Equations	---	---	---	4(1)	6(1)	10(2)
Vector Algebra	---	2(1)	---	8(2)	---	10(3)
Three Dimensional Geometry	1(1)	---	---	---	6(1)	7(2)
Linear Programming	---	2(1)	---	4(1)	---	6(2)
Probability	---	2(1)	4(1)	4(1)	---	10(3)
Total	**4(4)**	**16(8)**	**4(1)**	**40(10)**	**36(6)**	**100(29)**

* It is choice based.

Time Allowed : 3 hours **Maximum Marks : 100**

GENERAL INSTRUCTIONS

(i) All questions are compulsory.
(ii) Please check that this question paper contains **29** questions.
(iii) Questions **1-4** in Section-A are very short-answer type questions carrying **1** mark each.
(iv) Questions **5-12** in Section-B are short-answer type questions carrying **2** marks each.
(v) Questions **13-23** in Section-C are long-answer **I** type questions carrying **4** marks each.
(vi) Questions **24-29** in Section-D are long-answer **II** type questions carrying **6** marks each.
(vii) Please write down the serial number of the question before attempting it.

SECTION-A

1. If A is a 3×3 invertible matrix, then what will be the value of k if $\det(A^{-1}) = (\det A)^k$.

2. Determine the value of the constant 'k' so that the function $f(x)=\begin{cases} \frac{kx}{|x|}, & \text{if } x<0 \\ 3, & \text{if } x \geq 0 \end{cases}$ is continuous at $x = 0$.

3. Evaluate : $\int_2^3 3^x \, dx$.

4. If a line makes angles 90° and 60° respectively with the positive directions of x and y axes, find the angle which it makes with the positive direction of z-axis.

SECTION-B

5. Show that all the diagonal elements of a skew symmetric matrix are zero.

6. Find $\frac{dy}{dx}$ at $x = 1$, $y = \frac{\pi}{4}$ if $\sin^2 y + \cos xy = K$.

7. The volume of a sphere is increasing at the rate of 3 cubic centimeter per second. Find the rate of increase of its surface area, when the radius is 2 cm.

8. Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ is always increasing on R.

9. Find the vector equation of the line passing through the point $A(1, 2, -1)$ and parallel to the line $5x - 25 = 14 - 7y = 35z$.

10. Prove that if E and F are independent events, then the events E' and F' are also independent.

11. A small firm manufactures necklaces and bracelets. The total number of necklaces and bracelets that it can handle per day is at most 24. It takes one hour to make a bracelet and half an hour to make a necklace. The maximum number of hours available per day is 16. If the profit on a necklace is ₹ 100 and that on a bracelet is ₹ 300. Formulate an L.P.P for finding how many of each should be produced daily to maximize the profit ? It is being given that at least one of each must be produced.

12. Find $\int \frac{dx}{x^2 + 4x + 8}$.

SECTION-C

13. Prove that $\tan\left\{\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right\}$

$$+\tan\left\{\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right\} = \frac{2b}{a}.$$

14. Using properties of determinants, prove that

$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y).$$

OR

Let $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 2 \\ 7 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}$, find a matrix D such that $CD - AB = O$.

15. Differentiate the function $(\sin x)^x + \sin^{-1}\sqrt{x}$ with respect to x.

OR

If $x^m y^n = (x + y)^{m+n}$, prove that $\frac{d^2y}{dx^2} = 0$.

16. Find $\int \frac{2x}{(x^2+1)(x^2+2)^2}dx$

17. Evaluate : $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}dx$

OR

Evaluate : $\int_0^{3/2} |x \sin \pi x|\, dx$

18. Prove that $x^2 - y^2 = C(x^2 + y^2)^2$ is the general solution of the differential equation $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)\, dy$, where C is a parameter.

19. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then
    (a) Let $c_1 = 1$ and $c_2 = 2$, find c_3 which makes $\vec{a}$, $\vec{b}$ and $\vec{c}$ coplanar.
    (b) If $c_2 = -1$ and $c_3 = 1$, show that no value of c_1 can make $\vec{a}$, $\vec{b}$ and $\vec{c}$ coplanar.

20. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a} + \vec{b} + \vec{c}$ is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$. Also, find the angle which $\vec{a} + \vec{b} + \vec{c}$ makes with $\vec{a}$ or $\vec{b}$ or $\vec{c}$.

21. The random variable X can take only the values 0, 1, 2, 3. Given that $P(X = 0) = P(X = 1) = p$ and $P(X = 2) = P(X = 3)$ such that $\Sigma p_i x_i^2 = 2\Sigma p_i x_i$, find the value of p.

22. Often it is taken that a truthful person commands, more respect in the society. A man is known to speak the truth 4 out of 5 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.
    Do you also agree that the value of truthfulness leads to more respect in the society?

23. Solve the following L.P.P. graphically :
    Minimise $Z = 5x + 10y$
    Subject to constraints $x + 2y \le 120$, $x + y \ge 60$, $x - 2y \ge 0$ and $x, y \ge 0$

SECTION-D

24. Use product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$ to solve the system of equations $x + 3z = 9$, $-x + 2y - 2z = 4$, $2x - 3y + 4z = -3$

25. Consider $f: R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \left(\frac{\sqrt{y+6}-1}{3}\right)$. Hence find

(i) $f^{-1}(10)$ (ii) y if $f^{-1}(y) = \frac{4}{3}$

where R_+ is the set of all non-negative real numbers.

OR

Discuss the commutativity and associativity of binary operation '$*$' defined on $A = Q - \{1\}$ by the rule $a * b = a - b + ab$ for all $a, b \in A$. Also find the identity element of $*$ in A and hence find the invertible elements of A.

26. If the sum of lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum, when the angle between them is $\frac{\pi}{3}$.

27. Using integration, find the area of region bounded by the triangle whose vertices are (–2, 1), (0, 4) and (2, 3).

OR

Find the area bounded by the circle $x^2 + y^2 = 16$ and the line $\sqrt{3}y = x$ in the first quadrant, using integration.

28. Solve the differential equation $x\frac{dy}{dx} + y = x\cos x + \sin x$, given that $y = 1$ when $x = \frac{\pi}{2}$.

29. Find the equation of the plane through the line of intersection of $\vec{r}\cdot(2\hat{i} - 3\hat{j} + 4\hat{k}) = 1$ and $\vec{r}\cdot(\hat{i} - \hat{j}) + 4 = 0$ and perpendicular to the plane $\vec{r}\cdot(2\hat{i} - \hat{j} + \hat{k}) + 8 = 0$. Hence find whether the plane thus obtained contains the line $x - 1 = 2y - 4 = 3z - 12$.

OR

Find the vector and cartesian equations of a line passing through (1, 2, –4) and perpendicular to the two lines $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$.

SOLUTIONS

1. Given that, $\det(A^{-1}) = (\det A)^k$

i.e., $|A^{-1}| = |A|^k$

We know that $|A^{-1}| = \frac{1}{|A|} = |A|^{-1}$

$\therefore \quad k = -1$

2. We have, $f(x) = \begin{cases} \frac{kx}{|x|}, & x < 0 \\ 3, & x \ge 0 \end{cases}$

L.H.L. $= \lim_{x\to 0^-} f(x) = \lim_{x\to 0} \frac{kx}{-x} = -k$

R.H.L. $= \lim_{x\to 0^+} f(x) = \lim_{x\to 0} 3 = 3$

Since, $f(x)$ is continuous at $x = 0$.

$\therefore$ L.H.L. = R.H.L. $= f(0)$

$\Rightarrow -k = 3 \Rightarrow k = -3$

3. We have, $\int_2^3 3^x\,dx = \left[\frac{3^x}{\log 3}\right]_2^3$

$= \frac{3^3 - 3^2}{\log 3} = \frac{18}{\log 3}$

4. Let the line makes an angle α, β, γ with the positive direction of x, y, z axes respectively.

$\therefore \quad \alpha = 90°, \beta = 60°$

Now, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \cos^2 90° + \cos^2 60° + \cos^2\gamma = 1$

$\Rightarrow \cos^2\gamma = 1 - 0 - \frac{1}{4} = \frac{3}{4}$

$\Rightarrow \cos\gamma = \frac{\sqrt{3}}{2} \Rightarrow \gamma = 30°$

5. Let $A = [a_{ij}]$ be a skew symmetric matrix

Then, $a_{ij} = -a_{ji}\ \forall\ i, j$

$\Rightarrow a_{ii} = -a_{ii}\ \forall\ i$

$\Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0\ \forall\ i$

$\Rightarrow a_{11} = a_{22} = a_{33} = \ldots\ldots = a_{nn} = 0$

6. We have, $\sin^2 y + \cos xy = K$

Differentiating w.r.t. x, we get

$$2\sin y\cos y\frac{dy}{dx} + (-\sin xy)\left(x\frac{dy}{dx} + y\right) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy}$$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{\left(1, \frac{\pi}{4}\right)} = \frac{\frac{\pi}{4}\sin\frac{\pi}{4}}{\sin\frac{\pi}{2} - \sin\frac{\pi}{4}} = \frac{\pi}{4(\sqrt{2}-1)}$$

7. Let r, S and V respectively be the radius, surface area and volume of sphere at any time t.

Then, $\frac{dV}{dt} = 3\ \text{cm}^3/\text{sec}$

Now, $V=\frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$

$\Rightarrow \quad \frac{dr}{dt}=\frac{3}{4\pi r^2}$

Also, $S = 4\pi r^2 \Rightarrow \frac{dS}{dt}=8\pi r\frac{dr}{dt}$

$\Rightarrow \quad \frac{dS}{dt}=\frac{6}{r}=\frac{6}{2}=3\ \text{cm}^2/\text{sec}$

8. We have, $f(x) = 4x^3 - 18x^2 + 27x - 7$

$\Rightarrow \quad f'(x) = 12x^2 - 36x + 27$

$=12\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)+27$

$=12\left(x-\frac{3}{2}\right)^2-27+27=12\left(x-\frac{3}{2}\right)^2 \geq 0\ \forall\, x\in R$

Hence, $f(x)$ is always increasing on R.

9. Vector equation of the line passing through (1, 2, –1) and parallel to the line

$5x - 25 = 14 - 7y = 35z$

i.e., $\frac{x-5}{1/5}=\frac{y-2}{-1/7}=\frac{z}{1/35}$ or $\frac{x-5}{7}=\frac{y-2}{-5}=\frac{z}{1}$

is $\vec{r}=(\hat{i}+2\hat{j}-\hat{k})+\lambda(7\hat{i}-5\hat{j}+\hat{k})$

10. Since, E and F are independent events.

$\therefore \quad P(E\cap F) = P(E)\, P(F)$ ...(i)

Now, $P(E'\cap F') = 1 - P(E\cup F)$

$[\because P(E'\cap F') = P((E\cup F)')]$

$= 1 - [P(E) + P(F) - P(E\cap F)]$

$= 1 - P(E) - P(F) + P(E)\, P(F)$ [Using (i)]

$= (1 - P(E))\,(1 - P(F))$

$= P(E')\, P(F')$

Hence, E' and F' are independent events.

11. Let x necklaces and y bracelets be manufactured per day to maximize the profit.

$\therefore$ Maximize $Z = 100x + 300y$

Subject to the constraints : $x + y \leq 24$,

$(1)x+\left(\frac{1}{2}\right)y\leq 16 \Rightarrow 2x+y\leq 32$

and $x \geq 1, y \geq 1 \Rightarrow x - 1 \geq 0$ and $y - 1 \geq 0$

12. We have, $\int\frac{dx}{x^2+4x+8}=\int\frac{dx}{x^2+4x+4+4}$

$=\int\frac{dx}{(x+2)^2+(2)^2}=\frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right)+C$

13. We have, L.H.S. =

$$\tan\left\{\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right\}+\tan\left\{\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right\}$$

Let $\cos^{-1}\frac{a}{b}=\theta \Rightarrow \frac{a}{b}=\cos\theta$

$\therefore \quad$ L.H.S. $=\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)+\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$

$$=\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}+\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$$

$$=\frac{1+\tan^2\frac{\theta}{2}+1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}=2\left(\frac{1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}\right)$$

$$=\frac{2}{\cos\left(2\cdot\frac{\theta}{2}\right)}=\frac{2}{\cos\theta}=\frac{2}{a/b}=\frac{2b}{a}=\text{R.H.S.}$$

Hence proved.

14. L.H.S. $=\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x\end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} 3x+3y & x+y & x+2y \\ 3x+3y & x & x+y \\ 3x+3y & x+2y & x\end{vmatrix}$$

Taking $3(x + y)$ common from C_1, we get

$$3(x+y)\begin{vmatrix} 1 & x+y & x+2y \\ 1 & x & x+y \\ 1 & x+2y & x\end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_2$, we get

$$3(x+y)\begin{vmatrix} 1 & x+y & x+2y \\ 0 & -y & -y \\ 0 & 2y & -y\end{vmatrix}$$

Taking y common from R_2 and R_3 both, we get

$$3(x+y)\cdot y\cdot y\begin{vmatrix} 1 & x+y & x+2y \\ 0 & -1 & -1 \\ 0 & 2 & -1\end{vmatrix}$$

$= 3y^2\,(x + y)\cdot 1(1 + 2) = 9y^2\,(x + y) =$ R.H.S.

OR

We have, $A=\begin{bmatrix}2 & -1\\ 3 & 4\end{bmatrix}$, $B=\begin{bmatrix}5 & 2\\ 7 & 4\end{bmatrix}$, $C=\begin{bmatrix}2 & 5\\ 3 & 8\end{bmatrix}$

Let, $D=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

Now, $CD - AB = O$

$$\therefore \begin{bmatrix}2 & 5\\ 3 & 8\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}-\begin{bmatrix}2 & -1\\ 3 & 4\end{bmatrix}\begin{bmatrix}5 & 2\\ 7 & 4\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2a+5c & 2b+5d\\ 3a+8c & 3b+8d\end{bmatrix}-\begin{bmatrix}10-7 & 4-4\\ 15+28 & 6+16\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2a+5c-3 & 2b+5d\\ 3a+8c-43 & 3b+8d-22\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

On comparing the corresponding elements of the matrices, we get

$2a + 5c - 3 = 0$ ...(i) and $3a + 8c - 43 = 0$ ...(ii)

Also, $2b + 5d = 0$ ...(iii) and $3b + 8d - 22 = 0$ ...(iv)

Solving (i) and (ii), we get $a = -191,\ c = 77$

Solving (iii) and (iv), we get $b = -110,\ d = 44$

$$\therefore \quad D=\begin{bmatrix}-191 & -110\\ 77 & 44\end{bmatrix}$$

15. Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

$\Rightarrow \quad y = u + v$

[where $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$]

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad \text{...(i)}$$

Now, $u = (\sin x)^x$

Taking logarithm on both sides, we get

$\log u = x \log \sin x$

$$\Rightarrow \frac{1}{u}\frac{du}{dx}=x\frac{1}{\sin x}(\cos x)+\log \sin x$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x \ (x\cot x + \log \sin x) \qquad \text{...(ii)}$$

$v = \sin^{-1}\sqrt{x}$

$$\Rightarrow \frac{dv}{dx}=\left(\frac{1}{\sqrt{1-x}}\right)\cdot\frac{1}{2\sqrt{x}} \qquad \text{...(iii)}$$

From (i), (ii) and (iii), we get

$$\frac{dy}{dx} = (\sin x)^x\,(x\cot x + \log \sin x) + \frac{1}{2\sqrt{x}}\left(\frac{1}{\sqrt{1-x}}\right)$$

OR

We have, $x^m y^n = (x + y)^{m+n}$ ...(i)

Differentiating w.r.t. x, we get

$$mx^{m-1}y^n + nx^m y^{n-1}\frac{dy}{dx} = (m+n)(x+y)^{m+n-1}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow x^m y^n\left(\frac{m}{x}+\frac{n}{y}\frac{dy}{dx}\right)=\frac{(m+n)(x+y)^{m+n}}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \left(\frac{n}{y}-\frac{m+n}{x+y}\right)\frac{dy}{dx}=\frac{m+n}{x+y}-\frac{m}{x} \qquad \text{[Using (i)]}$$

$$\Rightarrow \left(\frac{nx+ny-my-ny}{y(x+y)}\right)\frac{dy}{dx}=\left(\frac{mx+nx-mx-my}{x(x+y)}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{x}\left(\frac{nx-my}{nx-my}\right)=\frac{y}{x} \qquad \text{...(ii)}$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\frac{x\frac{dy}{dx}-y}{x^2}=\frac{x\left(\frac{y}{x}\right)-y}{x^2}=0 \qquad \text{[Using (ii)]}$$

Hence proved.

16. Let $I=\int\frac{2x}{(x^2+1)(x^2+2)^2}dx$

Put $x^2 = y \Rightarrow 2x\,dx = dy$

$$\therefore \quad I=\int\frac{dy}{(y+1)(y+2)^2}$$

Let $\frac{1}{(y+1)(y+2)^2}=\frac{A}{y+1}+\frac{B}{y+2}+\frac{C}{(y+2)^2}$

$\Rightarrow \quad 1 = A(y + 2)^2 + B(y + 1)(y + 2) + C(y + 1)$

For $y = -1,\ 1 = A$

For $y = -2,\ 1 = -C \Rightarrow C = -1$

For $y = 0,\quad 1 = 4A + 2B + C \Rightarrow B = \frac{1-4+1}{2}=-1$

$$\therefore \quad I=\int\left[\frac{1}{y+1}-\frac{1}{y+2}-\frac{1}{(y+2)^2}\right]dy$$

$$= \log(y+1) - \log(y+2) + \frac{1}{y+2}+c$$

$$= \log\left(\frac{y+1}{y+2}\right)+\frac{1}{y+2}+c$$

$$= \log\left(\frac{x^2+1}{x^2+2}\right)+\frac{1}{x^2+2}+c \qquad [\because y = x^2]$$

17. Let $I=\int_0^{\pi}\left(\frac{x\sin x}{1+\cos^2 x}\right)dx$ ...(i)

$$\Rightarrow \quad I=\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx$$

$$\left[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$\Rightarrow \quad I=\int_0^{\pi}\left(\frac{(\pi-x)\sin x}{1+\cos^2 x}\right)dx \quad ...(ii)$$

Adding (i) and (ii), we get

$$2I=\int_0^{\pi}\frac{\pi\sin x}{1+\cos^2 x}dx$$

Put, $\cos x = t \Rightarrow -\sin x\, dx = dt$

when $x = 0$, $t = 1$ and $x = \pi$, $t = -1$

$$\therefore \quad 2I=\int_1^{-1}\frac{-\pi dt}{1+t^2}=\int_{-1}^{1}\frac{\pi dt}{1+t^2}$$

$$\Rightarrow \quad I=\frac{1}{2}\cdot 2\int_0^1\frac{\pi dt}{1+t^2}=\pi[\tan^{-1}t]_0^1$$

$$=\pi\left(\frac{\pi}{4}-0\right)=\frac{\pi^2}{4}$$

OR

Let $I=\int_0^{3/2}|x\sin\pi x|dx$

$$=\int_0^1 x\sin\pi x\,dx-\int_1^{3/2}x\sin\pi x\,dx$$

$$=\left[\frac{-x\cos\pi x}{\pi}\right]_0^1+\int_0^1\frac{\cos\pi x}{\pi}dx$$

$$+\left[\frac{x\cos\pi x}{\pi}\right]_1^{3/2}-\int_1^{3/2}\frac{\cos\pi x}{\pi}dx$$

$$=\frac{1}{\pi}-0+\left[\frac{\sin\pi x}{\pi^2}\right]_0^1+0-\frac{(-1)}{\pi}-\left[\frac{\sin\pi x}{\pi^2}\right]_1^{3/2}$$

$$=\frac{2}{\pi}+0-\frac{(-1)}{\pi^2}+0=\frac{2}{\pi}+\frac{1}{\pi^2}$$

18. We have, $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{x^3-3xy^2}{y^3-3x^2y} \quad ...(i)$$

Put, $y = vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$\therefore$ (i) becomes

$$v+x\frac{dv}{dx}=\frac{1-3v^2}{v^3-3v}$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{1-3v^2-v^4+3v^2}{v^3-3v}=\frac{1-v^4}{v(v^2-3)}$$

$$\Rightarrow \quad \frac{v(v^2-3)dv}{1-v^4}=\frac{dx}{x}$$

$$\Rightarrow \quad \int\frac{(v^3-3v)dv}{(1-v^2)(1+v^2)}=\int\frac{dx}{x} \quad ...(ii)$$

Now, let $\frac{v^3-3v}{(1-v^2)(1+v^2)}=\frac{Av+B}{1-v^2}+\frac{Cv+D}{1+v^2}$ ...(iii)

$\Rightarrow \quad v^3 - 3v = (Av + B)(1 + v^2) + (Cv + D)(1 - v^2)$

Comparing coeff. of like powers, we get

$A - C = 1$, $A + C = -3$, $B - D = 0$ and $B + D = 0$

Solving these equations, we get $A = -1$, $B = 0$, $C = -2$, $D = 0$

From (ii) and (iii), we have

$$\int\frac{-v}{1-v^2}dv-\int\frac{2v}{1+v^2}dv=\int\frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\log(1-v^2)-\log(1+v^2)=\log x+\log C_1$$

$$\Rightarrow \quad \frac{\sqrt{1-v^2}}{1+v^2}=C_1x$$

$$\Rightarrow \quad x\frac{\left(\sqrt{x^2-y^2}\right)}{x^2+y^2}=C_1x$$

$$\Rightarrow \quad x^2-y^2=C_1^2(x^2+y^2)^2$$

i.e., $x^2 - y^2 = C(x^2 + y^2)^2$ (where $C_1^2 = C$)

which is the required solution.

19. We have, $\vec{a}=\hat{i}+\hat{j}+\hat{k},\ \vec{b}=\hat{i},\ \vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$

(a) $c_1 = 1, c_2 = 2 \quad \therefore \quad \vec{c}=\hat{i}+2\hat{j}+c_3\hat{k}$

Given that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar

$$\therefore \quad \begin{vmatrix}1 & 1 & 1\\ 1 & 0 & 0\\ 1 & 2 & c_3\end{vmatrix}=0$$

$\Rightarrow \quad -1(c_3) + 1(2) = 0 \Rightarrow c_3 = 2$

(b) $c_2 = -1, c_3 = 1, \therefore \ \vec{c} = c_1\hat{i} - \hat{j} + \hat{k}$

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar

$$\therefore \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ c_1 & -1 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad -1(1) + 1(-1) = 0 \Rightarrow -2 = 0$, which is false.

So, no value of c_1 can make $\vec{a}$, $\vec{b}$ and $\vec{c}$ coplanar.

20. $|\vec{a}| = |\vec{b}| = |\vec{c}|$ (Given) ...(i)

and $\vec{a}\cdot\vec{b} = 0, \vec{b}\cdot\vec{c} = 0, \vec{c}\cdot\vec{a} = 0$ ...(ii)

Let $(\vec{a}+\vec{b}+\vec{c})$ be inclined to vectors $\vec{a}, \vec{b}, \vec{c}$ by angles α, β and γ respectively. Then

$$\cos\alpha = \frac{(\vec{a}+\vec{b}+\vec{c})\cdot\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} = \frac{\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}$$

$$= \frac{|\vec{a}|^2+0+0}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} \qquad [\text{Using (ii)}]$$

$$= \frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|} \qquad \text{...(iii)}$$

Similarly, $\cos\beta = \dfrac{|\vec{b}|}{|\vec{a}+\vec{b}+\vec{c}|}$ ...(iv)

and $\cos\gamma = \dfrac{|\vec{c}|}{|\vec{a}+\vec{b}+\vec{c}|}$ ...(v)

From (i), (iii), (iv) and (v), we get

$\Rightarrow \cos\alpha = \cos\beta = \cos\gamma \Rightarrow \alpha = \beta = \gamma$

Hence, the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to the vector $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Also the angle between them is given as

$$\alpha = \cos^{-1}\left(\frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|}\right), \ \beta = \cos^{-1}\left(\frac{|\vec{b}|}{|\vec{a}+\vec{b}+\vec{c}|}\right),$$

$$\gamma = \cos^{-1}\left(\frac{|\vec{c}|}{|\vec{a}+\vec{b}+\vec{c}|}\right)$$

21. We have, $P(X = 0) = P(X = 1) = p$

Let $P(X = 2) = P(X = 3) = k$

Since, X is a random variable taking values 0, 1, 2, 3

$\therefore \ P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = 1$

$\Rightarrow p + p + k + k = 1 \Rightarrow 2p + 2k = 1 \Rightarrow p + k = \dfrac{1}{2}$ ...(i)

Now, $\Sigma p_i x_i^2 = 2\Sigma p_i x_i$

$\Rightarrow \quad p(0) + p(1) + k(4) + k(9) = 2[p(0) + p(1) + k(2) + k(3)]$

$\Rightarrow \quad p + 13k = 2p + 10k$

$\Rightarrow \quad p - 3k = 0$ ...(ii)

Subtracting (ii) from (i), we get

$$4k = \frac{1}{2} \Rightarrow k = \frac{1}{8}$$

$\therefore$ From (i), we get $p = \dfrac{1}{2} - \dfrac{1}{8} = \dfrac{3}{8}$

22. Let E_1 be the event that '6' occurs, E_2 be the event that '6' does not occur and A be the event that the man reports that it is '6'.

$\therefore \quad P(E_1) = \dfrac{1}{6}, \ P(E_2) = \dfrac{5}{6}$

Now, $P(A/E_1)$ be the probability that the man reports that there is '6' on the die and '6' actually occurs

= Probability that the man speaks the truth $= \dfrac{4}{5}$

And $P(A/E_2)$ be the probability that the man reports that there is '6' when actually '6' does not occurs

= Probability that man does not speaks the truth

$$= 1 - \frac{4}{5} = \frac{1}{5}$$

$\therefore$ Required probability $= P(E_1/A)$

$$= \frac{P(E_1)\cdot P(A/E_1)}{P(E_1)\cdot P(A/E_1) + P(E_2)\cdot P(A/E_2)}$$

$$= \frac{\frac{1}{6}\times\frac{4}{5}}{\frac{1}{6}\times\frac{4}{5}+\frac{5}{6}\times\frac{1}{5}} = \frac{4}{4+5} = \frac{4}{9}$$

Yes, we are agree that the value of truthfulness leads to more respect in the society.

23. We have, Minimise $Z = 5x + 10y$

subject to constraints :

$x + 2y \le 120$

$x + y \ge 60$

$x - 2y \ge 0$

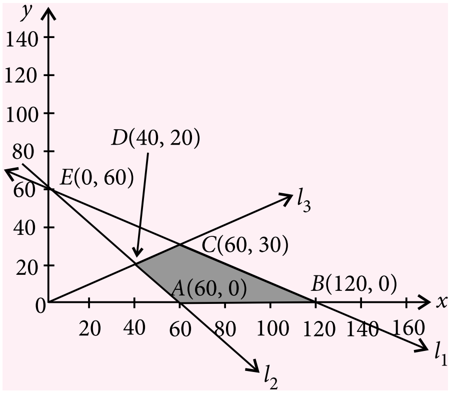
and $x, y \ge 0$

To solve L.P.P graphically, we convert inequations into equations.

$l_1 : x + 2y = 120$, $l_2 : x + y = 60$, $l_3 : x - 2y = 0$ and $x = 0$, $y = 0$

l_1 and l_2 intersect at $E(0, 60)$, l_1 and l_3 intersect at $C(60, 30)$, l_2 and l_3 intersect at $D(40, 20)$.

The shaded region $ABCD$ is the feasible region and is bounded. The corner points of the feasible region are $A(60, 0)$, $B(120, 0)$, $C(60, 30)$ and $D(40, 20)$.



Corner points	Value of $Z = 5x + 10y$
$A(60, 0)$	300 ← (Minimum)
$B(120, 0)$	600
$C(60, 30)$	600
$D(40, 20)$	400

Hence, Z is minimum at $A(60, 0)$ *i.e.*, 300.

24. We have, $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

$$=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=I$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1}=\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

$$\text{or} \quad \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & -2 \\ 2 & -3 & 4 \end{bmatrix}^{-1}=\begin{bmatrix} -2 & 9 & 6 \\ 0 & 2 & 1 \\ 1 & -3 & -2 \end{bmatrix}$$

Now the given system of equations is

$x + 3z = 9$

$-x + 2y - 2z = 4$

$2x - 3y + 4z = -3$

The system of equations can be written as $AX = B$

$$\text{where, } A=\begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & -2 \\ 2 & -3 & 4 \end{bmatrix}, X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}, B=\begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix}$$

Since, A^{-1} exists, so system of equations has a unique solution, given by

$X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -2 & 9 & 6 \\ 0 & 2 & 1 \\ 1 & -3 & -2 \end{bmatrix}\begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix}$$

$$=\begin{bmatrix} -18+36-18 \\ 8-3 \\ 9-12+6 \end{bmatrix}=\begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

$\Rightarrow \quad x = 0, y = 5, z = 3$

25. We have $f : R_+ \to [-5, \infty)$ is given by

$f(x) = 9x^2 + 6x - 5$

Let $x_1, x_2 \in R_+$ such that $f(x_1) = f(x_2)$

$\Rightarrow \quad 9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$

$\Rightarrow \quad 9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$

$\Rightarrow \quad (x_1 - x_2)(9(x_1 + x_2) + 6) = 0$

$\Rightarrow \quad x_1 = x_2 \text{ or } x_1 + x_2 = -\frac{6}{9}$

(which is not possible as $x_1, x_2 \in R_+$)

$\Rightarrow \quad f$ is one – one.

Let $\quad y = f(x) \ \forall\ y \in [-5, \infty)$

$\Rightarrow \quad 9x^2 + 6x - 5 = y \Rightarrow (3x + 1)^2 - 1 - 5 = y$

$\Rightarrow \quad 3x + 1 = \sqrt{y+6} \quad \Rightarrow \quad x = \frac{\sqrt{y+6}-1}{3}$

Now, x is defined and $x \in R_+$ if $y + 6 \ge 1 \Rightarrow y \ge -5$

$\Rightarrow \quad f$ is onto

$\therefore \quad f$ is one-one and onto.

$\Rightarrow \quad f$ is invertible and f^{-1} exists.

$$f^{-1}(y)=\frac{\sqrt{y+6}-1}{3}$$

(i) $\quad f^{-1}(10)=\frac{\sqrt{10+6}-1}{3}=\frac{4-1}{3}=1$

(ii) $\quad f^{-1}(y)=\frac{4}{3}=x$

$\therefore \quad y=f(x)=f\left(\frac{4}{3}\right)=9\left(\frac{4}{3}\right)^2+6\left(\frac{4}{3}\right)-5$

$= 16 + 8 - 5 = 19$

OR

We have, $a * b = a - b + ab \ \forall\ a, b \in A$,

where $A = Q - \{1\}$

Commutativity : Let $a, b \in Q - \{1\}$

We have, $a * b = a - b + ab \ne b - a + ab = b * a$

Hence, $*$ is not commutative.

Associativity : Let $a\ b, c, \in Q - \{1\}$

We have, $a * (b * c) = a * (b - c + bc)$

$= a - (b - c + bc) + (ab - ac + abc)$

$= a - b + c - bc + ab - ac + abc$

And $(a * b) * c = (a - b + ab) * c$

$= a - b + ab - c + ac - bc + abc$

$\therefore \quad a * (b * c) \neq (a * b) * c$

Hence, $*$ is not associative

Identity : Let e be the identity element in A.

$\therefore \quad a * e = a = e * a$

$\Rightarrow \quad a - e + ae = e - a + ea$

$\Rightarrow \quad a - e = e - a \Rightarrow e = a$

which is not possible, because identity should be unique element.

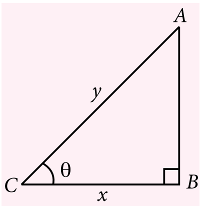
Hence, inverse of the element does not exist.

26. Let ABC be a right angled triangle with $BC = x$, $AC = y$ such that $x + y = k$, where k is any constant.

Let θ be the angle between the base and the hypotenuse.

Let P be the area of the triangle.

$$P = \frac{1}{2} \times BC \times AB = \frac{1}{2} \times x\, \sqrt{y^2 - x^2}$$



$$\Rightarrow \quad P^2 = \frac{x^2}{4}(y^2 - x^2)$$

$$\Rightarrow \quad P^2 = \frac{x^2}{4}[(k - x)^2 - x^2]$$

$$\Rightarrow \quad P^2 = \frac{k^2x^2 - 2kx^3}{4}$$

Let $Q = P^2$ *i.e.* $Q = \dfrac{k^2x^2 - 2kx^3}{4}$

$\therefore \quad P$ is maximum when Q is maximum.

Differentiating Q w.r.t. x, we get

$$\frac{dQ}{dx} = \frac{2k^2x - 6kx^2}{4} \qquad \text{...(i)}$$

For maximum or minimum area,

$$\frac{dQ}{dx} = 0 \Rightarrow \quad k^2x - 3kx^2 = 0 \quad \Rightarrow \quad x = \frac{k}{3}$$

Differentiating (i) w.r.t. x, we get

$$\frac{d^2Q}{dx^2} = \frac{2k^2 - 12kx}{4}$$

$$\therefore \quad \left[\frac{d^2Q}{dx^2}\right]_{x=\frac{k}{3}} = \frac{-k^2}{2} < 0$$

Thus, Q is maximum when $x = \dfrac{k}{3}$

$\Rightarrow \quad P$ is maximum at $x = \dfrac{k}{3}$

Now, $x = \dfrac{k}{3} \Rightarrow \ y = k - \dfrac{k}{3} = \dfrac{2k}{3} \qquad [\because \ x + y = k]$

$$\therefore \quad \cos\theta = \frac{x}{y} = \frac{k/3}{2k/3} = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$$

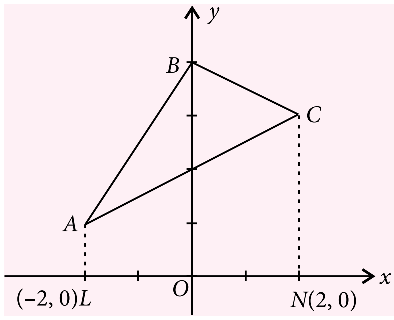
So, the area of ΔABC is maximum when angle between the hypotenuse and base is $\dfrac{\pi}{3}$.

27. Let $A(-2, 1)$, $B(0, 4)$ and $C(2, 3)$

Eq. of AB is $y - 4 = \left(\dfrac{4-1}{2}\right)(x - 0) \ \Rightarrow \ y = \dfrac{3}{2}x + 4$

Eq. of BC is $y - 3 = \left(\dfrac{3-4}{2}\right)(x - 2) \Rightarrow y = \dfrac{-1}{2}x + 4$

Eq. of AC is $y - 1 = \left(\dfrac{3-1}{2+2}\right)(x + 2) \Rightarrow y = \dfrac{x}{2} + 2$



Area of required region, = Area of trap. $ALOB$

+ Area of trap. $BONC$ – Area of trap. $ALNC$

$$= \int_{-2}^{0}\left(\frac{3}{2}x + 4\right)dx + \int_{0}^{2}\left(\frac{-1}{2}x + 4\right)dx - \int_{-2}^{2}\left(\frac{x}{2} + 2\right)dx$$

$$= \left[\frac{3}{4}x^2 + 4x\right]_{-2}^{0} + \left[\frac{-x^2}{4} + 4x\right]_{0}^{2} - \left[\frac{x^2}{4} + 2x\right]_{-2}^{2}$$

$= (-3 + 8) + (-1 + 8) - (5 + 3) = 4$ sq. units

OR

We have curves, $y = \dfrac{1}{\sqrt{3}}x$ ...(i)

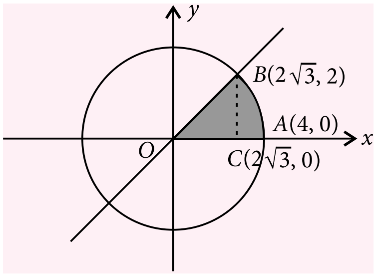
and $x^2 + y^2 = 16$ ...(ii)

Curves (i) and (ii) intersect at $(2\sqrt{3}, 2)$ and $(-2\sqrt{3}, -2)$.

$\therefore$ Required area = Area of region $OBAO$

= area ΔOBC + area of region $BCAB$

$$= \int_0^{2\sqrt{3}} \frac{x}{\sqrt{3}}dx + \int_{2\sqrt{3}}^{4} \sqrt{16-x^2}dx$$



$$= \left[\frac{x^2}{2\sqrt{3}}\right]_0^{2\sqrt{3}} + \left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_{2\sqrt{3}}^{4}$$

$$= 2\sqrt{3} + 8\left(\frac{\pi}{2}\right) - 2\sqrt{3} - \frac{8\pi}{3}$$

$$= \frac{12\pi - 8\pi}{3} = \frac{4\pi}{3} \text{ sq.units}$$

28. We have, $x\dfrac{dy}{dx} + y = x\cos x + \sin x$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = \cos x + \frac{\sin x}{x}$$

It is a linear differential equation.

$$\text{I.F.} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

$$\therefore\ y \cdot x = \int x\left(\cos x + \frac{\sin x}{x}\right)dx + c$$

$$= \int [x\cos x + \sin x]dx + c$$

$$= x\sin x - \int \sin x dx + \int \sin x dx + c$$

$= x$ sin$x$ + c

$$\Rightarrow\ y = \sin x + \frac{c}{x}$$

Given that, $y = 1$ when $x = \dfrac{\pi}{2}$

$$\therefore\ 1 = 1 + \frac{c}{\pi/2} \Rightarrow c = 0$$

$\therefore$ y = sin$x$ is the required solution.

29. The equation of any plane through the line of intersection of the given planes is

$$[\vec{r}\cdot(2\hat{i} - 3\hat{j} + 4\hat{k}) - 1] + \lambda[\vec{r}\cdot(\hat{i} - \hat{j}) + 4] = 0$$

$$\Rightarrow\ \vec{r}\cdot[(2+\lambda)\hat{i} + (-3-\lambda)\hat{j} + 4\hat{k}] = 1 - 4\lambda \qquad \text{...(i)}$$

If plane (i) is perpendicular to $\vec{r}\cdot(2\hat{i} - \hat{j} + \hat{k}) + 8 = 0$,

then $[(2+\lambda)\hat{i} + (-3-\lambda)\hat{j} + 4\hat{k}]\cdot[2\hat{i} - \hat{j} + \hat{k}] = 0$

$\Rightarrow$ $(2 + \lambda)2 + (-3 - \lambda)(-1) + 4(1) = 0$

$\Rightarrow$ $4 + 2\lambda + 3 + \lambda + 4 = 0$

$$\Rightarrow\ 3\lambda + 11 = 0 \Rightarrow \lambda = \frac{-11}{3}$$

Putting $\lambda = \dfrac{-11}{3}$ in (i), we obtain the equation of required plane *i.e.*,

$$\vec{r}\cdot\left[\left(2 - \frac{11}{3}\right)\hat{i} + \left(-3 + \frac{11}{3}\right)\hat{j} + 4\hat{k}\right] = 1 + \frac{44}{3}$$

$$\Rightarrow\ \vec{r}\cdot\left(-\frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + 4\hat{k}\right) = \frac{47}{3}$$

$$\Rightarrow\ \vec{r}\cdot(-5\hat{i} + 2\hat{j} + 12\hat{k}) = 47$$

Now given line is $x - 1 = 2y - 4 = 3z - 12$

i.e., $\dfrac{x-1}{1} = \dfrac{y-2}{\frac{1}{2}} = \dfrac{z-4}{\frac{1}{3}}$

or $\dfrac{x-1}{6} = \dfrac{y-2}{3} = \dfrac{z-4}{2}$

Now, the line passes through the point (1, 2, 4) satisfies the equation of plane. So, the plane contains the line.

OR

Let the equation of line passing through (1, 2, –4) and perpendicular to the lines

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \text{ and } \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$$

be $\dfrac{x-1}{l} = \dfrac{y-2}{m} = \dfrac{z+4}{n}$ ...(i)

$\therefore$ $l(3) + m(-16) + n(7) = 0$ and
$l(3) + m(8) + n(-5) = 0$

$$\Rightarrow\ \frac{l}{80-56} = \frac{m}{21+15} = \frac{n}{24+48}$$

$$\Rightarrow\ \frac{l}{24} = \frac{m}{36} = \frac{n}{72} \Rightarrow \frac{l}{2} = \frac{m}{3} = \frac{n}{6}$$

$\therefore$ The equation of the required line is

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$$

and its vector equation is

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

CLASS XI Series 1

ACE YOUR WAY CBSE

Sets

Important Formulae

- *Representation of Sets*
  - *In Roster Method or Listing Method or Tabular Method: We list all the elements of the set in a row by putting {} braces and separating them by commas.*
  - *In Set Builder Method : A set is represented by a characterizing property $q(x)$ of its elements x.*
- *A set whose elements can be counted is called a finite set and the set which is not finite is called an infinite set.*
- *Two sets A and B are called equal if A and B have identical elements.*
- *A set having no element in it is called a null set.*
- *Set A is said to be subset of set B if every member of set A is also the member of set B.*
- *If $A \subseteq B$, then B is called superset of A.*
- *If $A \subset B$ and $A \neq B$, then A is called proper subset of B.*
- *Set of all the subsets of a set A is called its power set which is denoted by $P(A)$.*
- *If all sets under consideration are subsets of a larger set, then this larger set is called universal set, denoted by U.*
- $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- $A \cap B = \{x : x \in A \text{ and } x \in B\}$
- *$A - B$ is the set of elements which belong to A but not to B.*
- $A' = \{x : x \in U \text{ but } x \notin A\}$
- *If A and B are any two sets, then*
  - $A - B = A \cap B'$
  - $B - A = B \cap A'$
  - $A - B = A \Leftrightarrow A \cap B = \phi$
  - $(A - B) \cup B = A \cup B$
  - $(A - B) \cap B = \phi$
  - $A \subseteq B \Leftrightarrow B' \subseteq A'$
  - $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
- *Commutative law*
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- *Associative law*
  - $(A \cup B) \cup C = A \cup (B \cup C)$
  - $(A \cap B) \cap C = A \cap (B \cap C)$
- *Law of identity element*
  - $A \cup \phi = A$
  - $\phi \cap A = \phi$
- *Idempotent law*
  - $A \cup A = A$
  - $A \cap A = A$
- *Distributive law*
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- *Laws of U*
  - $U \cup A = U$
  - $U \cap A = A$
- *Complement law*
  - $A \cup A' = U$
  - $A \cap A' = \phi$
- *De-Morgan's law*
  - $(A \cup B)' = A' \cap B'$
  - $(A \cap B)' = A' \cup B'$
- *Law of double complementation*
  - $(A')' = A$
- *Law of empty set and finite universal set*
  - $\phi' = U$ and $U' = \phi$
- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A - B) = n(A \cap B') = n(A) - n(A \cap B)$
- $n(A \Delta B) = n(A) + n(B) - 2n(A \cap B)$
- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

WORK IT OUT

VERY SHORT ANSWER TYPE

1. Write the set $A = \{x : x \in Z, x^2 < 20\}$ in the roster form.

2. From the sets given below, select equal set:
$A = \{2, 4, 8, 12\}$, $B = \{1, 2, 3, 4\}$,
$C = \{4, 8, 12, 14\}$ $D = \{3, 1, 4, 2\}$,
$E = \{-1, 1\}$, $F = \{0, a\}$
$G = \{1, -1\}$, $H = \{0, 1\}$

3. Write down all the subsets of the set $\{1, 2, 3\}$.

4. Let $U = \{1, 2, 3, 4, 5, 6, 8\}$, $A = \{2, 3, 4\}$, $B = \{3, 4, 5\}$. Show that $(A \cup B)' = A' \cap B'$ and $A' \cap B' = A' \cup B'$.

5. Represent the set $A = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}\right\}$ in set builder form.

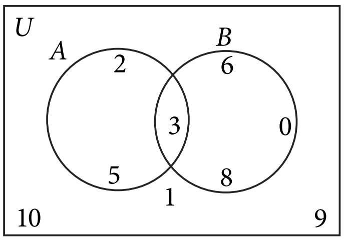
SHORT ANSWER TYPE

6. If $A = \{3, 5, 7, 9, 11\}$, $B = \{7, 9, 11, 13\}$, $C = \{11, 13, 15\}$ and $D = \{15, 17\}$, find:
(i) $A \cap (B \cup C)$ (ii) $(A \cup D) \cap (B \cup C)$

7. If A and B are two sets such that $n(A) = 27$, $n(B) = 35$ and $n(A \cup B) = 50$, find $n(A \cap B)$.

8. If $A \subseteq B$, prove that $(C - B) \subseteq (C - A)$.

9. From the adjoining Venn diagram, determine the following sets:



(i) $A \cup B$ (ii) $A \cap B$
(iii) $A - B$ (iv) $(A \cap B)'$

10. For any sets A and B, show that $(A - B) = (A \cap B')$.

LONG ANSWER TYPE - I

11. Two finite sets have m *and* n elements. The total number of subsets of the first set is 56 more than the total number of subsets of the second set. Find the value of m and n.

12. If $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 3, 5, 8\}$, $C = \{2, 5, 7, 8\}$, verify that $A - (B \cup C) = (A - B) \cap (A - C)$.

13. In a group of 800 people, 550 can speak Hindi and 450 can speak English. How many can speak both Hindi and English?

14. If A and B are respectively the sets having the elements as the zeros of the polynomials $x^3 - 4x^2 + x + 6$ and $x^3 - 6x^2 + 11x - 6$, find
(i) $A - B$ (ii) $B - A$
(iii) $A - (B - A)$

15. If A and B are two sets containing 3 and 6 elements respectively, what can be the maximum number of elements in $A \cup B$?
Find also the minimum number of elements in $A \cup B$.

LONG ANSWER TYPE - II

16. There are 200 individuals with a skin disorder, 120 had been exposed to the chemical C_1, 50 to chemical C_2 and 30 to both the chemicals C_1 and C_2. Find the number of individuals exposed to:
(i) Chemical C_2 but not chemical C_1.
(ii) Chemical C_1 or chemical C_2.

17. In a class, 18 students took Physics, 23 students took Chemistry and 24 students took Mathematics. Of these 13 took both Chemistry and Mathematics, 12 took both Physics and Chemistry, 11 took both Physics and Mathematics and 6 students offered all the three subjects. If each student took atleast one of the three subjects, then find :
(i) Total number of students in the class.
(ii) How many took Mathematics but not Chemistry?
(iii) How many took exactly one of the 3 subjects?

18. A school awarded 15 medals in table tennis, 38 in hockey and 20 in cricket. If these medals went to a total of 58 men and only three men got medals in all the three sports, how many received medals in exactly two of the three sports?

19. If A, B, C are three sets and U is the universal set such that $n(U) = 800$, $n(A) = 200$, $n(B) = 300$ and $n(A \cap B) = 100$. Find $n(A' \cap B')$.

20. There are 2000 students in a school. Out of these 1000 play cricket, 600 play basketball and 550 play football, 120 play cricket and basketball, 80 play basketball and football, 150 play cricket and football and 45 play all the three games. How many students play none of the games?

SOLUTIONS

1. We observe that the integers whose squares are less than 20 are: 0, ± 1, ± 2, ± 3, ± 4. Therefore, the set A in roster form is $A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$.

2. From the given sets, we see that sets B and D and also sets E and G have same elements.

$\therefore$ $B = D = \{1, 2, 3, 4\}$ and $E = G = \{-1, 1\}$ are equal sets.

3. Number of elements in given set = 3

Number of subsets of given set = $2^3 = 8$

$\therefore$ Subsets of given set are ϕ, {1}, {2}, {3}, {1, 2}, {2, 3}, {1, 3}, {1, 2, 3}.

4. Given, $U = \{1, 2, 3, 4, 5, 6, 8\}$

$A = \{2, 3, 4\}$, $B = \{3, 4, 5\}$

$A \cup B = \{2, 3, 4\} \cup \{3, 4, 5\} = \{2, 3, 4, 5\}$

$\therefore$ $(A \cup B)' = \{2, 3, 4, 5\}' = \{1, 6, 8\}$ ...(i)

$A' = \{2,3,4\}' = \{1, 5, 6, 8\}$

$B' = \{3, 4, 5\}' = \{1, 2, 6, 8\}$

$\therefore$ $A' \cap B' = \{1, 5, 6, 8\} \cap \{1, 2, 6, 8\}$

$= \{1, 6, 8\}$ ...(ii)

From (i) and (ii), we get

$(A \cup B)' = A' \cap B'$

Now, $A \cap B = \{2, 3, 4\} \cap \{3, 4, 5\} = \{3, 4\}$

$(A \cap B)' = \{1, 2, 5, 6, 8\}$ ...(iii)

$A' \cup B' = \{1, 5, 6, 8\} \cup \{1, 2, 6, 8\}$

$= \{1, 2, 5, 6, 8\}$ ...(iv)

From (iii) and (iv), we get, $(A \cap B)' = A' \cup B'$.

5. $A = \left\{\dfrac{n}{n+1}, n \in N, n \le 6\right\}$

6. (i) $A \cap (B \cup C)$

$= \{3, 5, 7, 9, 11\} \cap (\{7, 9, 11, 13\} \cup \{11, 13, 15\})$

$= \{3, 5, 7, 9, 11\} \cap \{7, 9, 11, 13, 15\} = \{7, 9, 11\}$

(ii) $(A \cup D) \cap (B \cup C)$

$= (\{3, 5, 7, 9, 11\} \cup \{15, 17\}) \cap (\{7, 9, 11, 13\} \cup \{11, 13, 15\})$

$= \{3, 5, 7, 9, 11, 15, 17\} \cap \{7, 9, 11, 13, 15\}$

$= \{7, 9, 11, 15\}$

7. We know that

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow$ $n(A \cap B) = n(A) + n(B) - n(A \cup B)$

$= (27 + 35 - 50) = 12.$

Hence, $n(A \cap B) = 12$.

8. Given, $A \subseteq B$

Let $x \in (C - B)$. Then, $x \in C$ and $x \notin B$

$\Rightarrow$ $x \in C$ and $x \notin A$ $\quad [\because A \subseteq B]$

$\Rightarrow$ $x \in (C - A)$

$\therefore$ $(C - B) \subseteq (C - A)$

9. (i) {0, 2, 3, 5, 6, 8} (ii) {3}

(iii) {2, 5} (iv) {0, 1, 2, 5, 6, 8, 9, 10}

10. Let $x \in A - B$. Then, $x \in A - B \Rightarrow x \in A$ and $x \notin B$

$\Rightarrow$ $x \in A$ and $x \in B'$ $\Rightarrow$ $x \in A \cap B'$.

$\therefore$ $(A - B) \subseteq (A \cap B')$ ...(i)

Again, let $y \in (A \cap B')$. Then,

$y \in (A \cap B') \Rightarrow y \in A$ and $y \in B'$

$\Rightarrow$ $y \in A$ and $y \notin B \Rightarrow y \in (A - B)$

$\therefore$ $(A \cap B') \subseteq (A - B)$. ...(ii)

From (i) and (ii), we get $(A - B) = A \cap B'$

11. Number of subsets of first set = 2^m

Number of subsets of second set = 2^n

$2^m - 2^n = 56 \Rightarrow 2^n(2^{m-n} - 1) = 2^3(2^3 - 1)$

$\Rightarrow$ $n = 3$ and $m - n = 3 \Rightarrow n = 3$ and $m = 6$.

12. We have $B \cup C = \{1, 2, 3, 5, 7, 8\}$

L.H.S. $= A - (B \cup C) = \{4\}$

We have $= A - B = \{2, 4\}$; $A - C = \{1, 3, 4\}$

R.H.S. $= (A - B) \cap (A - C) = \{4\}$

$\therefore$ L.H.S. = R.H.S.

13. Let H denote the set of people speaking Hindi and E denote the set of people speaking English.

Given : $n(H) = 550$, $n(E) = 450$ and $n(H \cup E) = 800$.

We have to find $n(H \cap E)$.

We know that

$n(H \cup E) = n(H) + n(E) - n(H \cap E)$

$\Rightarrow$ $n(H \cap E) = n(H) + n(E) - n(H \cup E)$

$= 550 + 450 - 800 = 200.$

Hence, 200 persons can speak both Hindi and English.

14. Zeros of $x^3 - 4x^2 + x + 6$ are $-1, 2, 3$ $\therefore$ $A = \{-1, 2, 3\}$ and Zeros of $x^3 - 6x^2 + 11x - 6$ are 1, 2, 3. $\therefore$ $B = \{1, 2, 3\}$.

(i) $A - B = \{-1\}$; (ii) $B - A = \{1\}$; (iii) $A - (B - A) = A - \{1\} = \{-1, 2, 3\}$

15. We know that

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$ ...(i)

Case 1 : From (i), it is clear that $n(A \cup B)$ will be maximum when $n(A \cap B) = 0$.

MPP-1 CLASS XI ANSWER KEY

1.	(b)	**2.**	(d)	**3.**	(a)	**4.**	(d)	**5.**	(a)
6.	(a)	**7.**	(a,c)	**8.**	(a,b)	**9.**	(b,c)	**10.**	(c)
11.	(c,d)	**12.**	(a,c,d)	**13.**	(a,b,c)	**14.**	(b)	**15.**	(a)
16.	(b)	**17.**	(4)	**18.**	(7)	**19.**	(3)	**20.**	(3)

In that case, $n(A \cup B) = n(A) + n(B) = (3 + 6) = 9$

$\therefore$ Maximum number of elements in $(A \cup B) = 9$.

Case 2 : From (i), it is clear that $n(A \cup B)$ will be minimum when $n(A \cap B)$ is maximum, *i.e.*, when $n(A \cap B) = 3$.

In this case, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
$= (3 + 6 - 3) = 6$

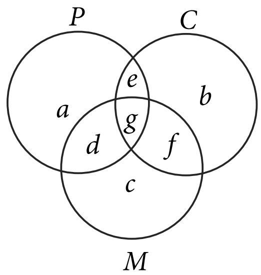
$\therefore$ Minimum number of elements in $A \cup B = 6$.

16. $n(U) = 200$, $n(C_1) = 120$, $n(C_2) = 50$
$n(C_1 \cap C_2) = 30$

(i) $n(C_2 \text{ but not } C_1) = n(C_2) - n(C_1 \cap C_2)$
$= 50 - 30 = 20$

(ii) $n(C_1 \text{ or } C_2) = n(C_1) + n(C_2) - n(C_1 \cap C_2)$
$= 120 + 50 - 30 = 140$

17. P : Physics; C : Chemistry; M : Mathematics



Given, $a + d + e + g = 18$; $f + g = 13$;
$b + e + f + g = 23$; $e + g = 12$;
$c + d + f + g = 24$; $d + g = 11$; $g = 6$

On solving above equations, we get
$g = 6, f = 7, e = 6, d = 5, a = 1, b = 4, c = 6$

(i) Total number of students in class
$= a + b + c + d + e + f + g = 35$

(ii) Mathematics but not Chemistry $= d + c = 11$

(iii) Exactly one of the three subjects $= a + b + c = 11$

18. Let T denote the set of men who received medals in table tennis, B the set of men who received medals in hockey and C the set of men who received medals in cricket. Then, we have

$n(T) = 15$, $n(H) = 38$, $n(C) = 20$, $n(T \cup H \cup C) = 58$ and $n(T \cap H \cap C) = 3$

Now, $n(T \cup H \cup C) = n(T) + n(H) + n(C) - n(T \cap H) - n(H \cap C) - n(T \cap C) + n(T \cap H \cap C)$

$\Rightarrow$ $58 = 38 + 15 + 20 - n(T \cap H) - n(H \cap C) - n(T \cap C) + 3$

$\Rightarrow$ $n(T \cap H) + n(H \cap C) + n(T \cap C) = 76 - 58 = 18$

Now, number of men who received medals in exactly two of the three sports

$= n(T \cap H) + n(H \cap C) + n(T \cap C) - 3n(T \cap H \cap C)$
$= 18 - 3 \times 3 = 9$

Thus, 9 men received medals in exactly two of the three sports.

19. We know, $n(A' \cap B') = n(A \cup B)'$ ...(i)

Now, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
$= 200 + 300 - 100$ (given)
$= 400$

$\therefore$ $n(A \cup B)' = n(U) - n(A \cup B)$
$= 800 - 400 = 400$

$\therefore$ $n(A' \cap B') = 400$ [From (i)]

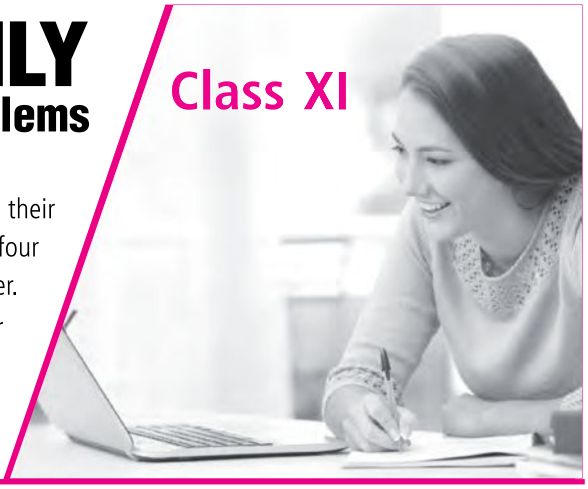
20. A : set of students playing cricket
B : set of students playing basketball
C : set of students playing football

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$
$= 1000 + 600 + 550 - 120 - 80 - 150 + 45 = 1845$

Number of students playing none of the games
$= 2000 - 1845 = 155$.

MPP-1 MONTHLY Practice Problems

Class XI



This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Sets, Relations and Functions

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. Find the domain of the function

$$f(x)=\frac{1}{\sqrt{[x]^2-[x]-6}}$$

(a) $(-\infty, -3) \cup [4, \infty)$
(b) $(-\infty, -2) \cup [4, \infty)$
(c) $(-\infty, 2] \cup (4, \infty)$
(d) $[4, \infty)$

2. If $a\,f(x)+b\,f\left(\frac{1}{x}\right)=x-1, x \neq 0$ and $a \neq b$, then $f(2)$ is equal to

(a) $\frac{a}{a^2-b^2}$ (b) $\frac{(a+2b)}{2(a^2-b^2)}$
(c) $\frac{(a-2b)}{(a^2-b^2)}$ (d) $\frac{(2a+b)}{2(a^2-b^2)}$

3. If $X = \{8^n - 7n - 1 : n \in N\}$ and $Y = \{49(n-1) : n \in N\}$ then

(a) $X \subseteq Y$ (b) $Y \subseteq X$
(c) $X = Y$ (d) none of these

4. The range of the function $f(x) = 9^x - 3^x + 1$ is

(a) $(-\infty, \infty)$ (b) $(-\infty, 0)$
(c) $(0, \infty)$ (d) $\left[\frac{3}{4}, \infty\right)$

5. Number of non empty subsets of the set consisting 10 elements in all, is

(a) 1023 (b) 1024
(c) $2^{10} - 2$ (d) 2^9

6. Let $S = \{x : x$ is a positive multiple of 3 less than 100$\}$ $P = \{x : x$ is a prime number less than 20$\}$. Then, one's digit of $(n(S) + n(P))$ is

(a) 1 (b) 4
(c) 2 (d) 3

One or More Than One Option(s) Correct Type

7. If $X \cup \{1, 2\} = \{1, 2, 3, 5, 9\}$, then

(a) the smallest set X is $\{3, 5, 9\}$
(b) the smallest set X is $\{2, 3, 5, 9\}$
(c) the largest set X is $\{1, 2, 3, 5, 9\}$
(d) the largest set X is $\{2, 3, 4, 9\}$

8. If $f(x)=\frac{x^2-9}{x-3}$ then for $f(x)$

(a) domain $= R - \{3\}$
(b) range $= R - \{6\}$
(c) domain $= (-\infty,2) \cup (4, \infty)$
(d) range $= R$

9. If $A = \{(x, y) : y = e^{2x}, x \in R\}$ and $B\,\{(x, y) : y = e^{-2x}, x \in R\}$, then

(a) $A \cap B = \phi$
(b) $A \cap B \neq \phi$

Solution Sender of Maths Musing

SET-172

1. V. Damodhar Reddy (Telangana)
2. Khokon Kumar Nandi (West Bengal)

SET-171

1. Satyadev. P (Bangalore)

(c) $A \cap B$ is a singleton set

(d) None of the above

10. If $A = \{x : f(x) = 0\}$ and $B = \{x : g(x) = 0\}$, then $A \cap B$ will be

(a) $\frac{f(x)}{g(x)}$ (b) $\frac{g(x)}{f(x)}$

(c) $[f(x)]^2 + [g(x)]^2 = 0$ (d) none of these

11. A real valued function $f(x)$ satisfies the functional equation $f(x - y) = f(x) f(y) - f(a - x) f(a + y)$ where a is a given constant and $f(0) = 1$, then,

(a) $f(a) = 1$

(b) $f(2a - x) = f(a) + f(a - x)$

(c) $f(a) = 0$

(d) $f(2a - x) = - f(x)$

12. Let $X = \{5, 6, \{1, 2\}, 7, 9\}$ then which of the following are subset (S) of X?

(a) $\{5, 6, 9\}$ (b) $\{1, 2, 7\}$

(c) $\{6, \{1, 2\}\}$ (d) $\{\{1, 2\}\}$

13. The domain of the function $f(x)=\sqrt{\frac{(x+1)(x-3)}{x-2}}$ contains

(a) $[0, 2) \cup [3, 100]$ (b) $(-1, 2) \cup [3, \infty)$

(c) $[-1, 1] \cup [5, \infty)$ (d) $(-1, 5)$

Comprehension Type

Out of 100 students; 15 passed in English, 12 passed in Mathematics, 8 in Science, 6 in English and Mathematics, 7 in Mathematics and Science, 4 in English and Science; 4 passed in all the three subjects.

14. The number of students passed in English and Mathematics but not in Science is

(a) 3. (b) 2

(c) 4 (d) 5

15. The number of students who only passed in more than one subject is

(a) 9 (b) 3

(c) 2 (d) 1

Matrix Match Type

16. Match the following :

	Column I		Column II
P.	$f(x) = \log_{10} \sin(x - 3) + \sqrt{16-x^2}$	1.	$x \in (0, \infty)$
Q.	$f(x)=\sqrt{\frac{4-\lvert x\rvert}{7-\lvert x\rvert}}$	2.	$x \in R - \{-1, 1\}$
R.	$f(x)=\frac{1}{\sqrt{x+[x]}}$	3.	$x\in\ ]-2\pi+3, -\pi+3\ [\cup]3, 4]$
S.	$f(x)=\frac{1}{1-x^2}$	4.	$x\in(-\infty,-7)\cup[-4,4] \cup (7, \infty)$

	P	Q	R	S
(a)	4	2	3	1
(b)	3	4	1	2
(c)	1	4	3	2
(d)	2	4	3	1

Integer Answer Type

17. If $A = \{3, 5, 7, 9, 11\}$, $B = \{7, 9, 11, 13\}$, $C = \{11, 13, 15\}$ and $D = \{15, 17\}$, then number of elements in $(A \cup D) \cap (B \cup C)$ is equal to

18. Let X be the universal set for sets A and B. If $n(A) = 200$, $n(B) = 300$, $n(A \cap B) = 100$, $n(A' \cap B') = 300$. If $n(X)$ is equal to 100 a then a equals

19. If $f(x) + 2 f(1 - x) = x^2 + 2\ \forall\ x \in R$, then $f(5)$ is equal to

20. Two finite sets with m, n elements, the total number of subsets of the first set is 224 more than the total number of subsets of the second. Then $m - n$ equals

◈◈

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ……

No. of questions correct ……

Marks scored in percentage ……

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

CLASS XII | Series 1

ACE YOUR WAY CBSE

Relations and Functions

Important Formulae

- If X and Y be two non-empty finite sets having m and n elements respectively then
  - (i) Number of ordered pairs in $X \times Y$ is $m \times n$
  - (ii) Number of subsets of $X \times Y = 2^{m \times n}$
  - (iii) Number of relations from X to $Y = 2^{mn}$
- Empty relation : A relation R in X given by $R = \phi \subset X \times X$.
- Universal relation : A relation R in X given by $R = X \times X$.
- Reflexive relation : A relation R in X with $(a, a) \in R$ $\forall\, a \in X$.
- Symmetric relation : A relation R in X with $(a, b) \in R$ implies $(b, a) \in R\ \forall\, a, b \in X$.
- Transitive relation : A relation R in X with $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R\ \forall\, a, b, c \in X$.
- Equivalence relation : A relation R in X which is reflexive, symmetric and transitive.
- If R is a relation on A, then Inverse relation on $A = R^{-1} = \{(b, a) : (a, b) \in R\}$
- Composition of Relation : Let $R \subseteq A \times B$, $S \subseteq B \times C$ be two relations, then composite relation of R and S is $SoR \subseteq A \times C$ or $SoR = \{(a, c) : (a, b) \in R, (b, c) \in S\}$
- The number of functions from a finite set A into set $B = [n(B)]^{[n(A)]}$
- There may exist some elements in set B which are not the images of any element in set A.
- If $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, for every $x_1, x_2 \in$ domain, then f is one-one or else many one.
- If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, for every $x_1, x_2 \in$ domain, then f is one-one or many one.
- If the range of the function equals to the codomain of the function, the function is onto.
- A function f is invertible if and only if f is one-one and onto.
- Inverse function : Inverse of a function $f : A \rightarrow B$ is defined by $f^{-1} : B \rightarrow A$
  $\therefore\quad f^{-1}(y) = x \Leftrightarrow f(x) = y$
- Composition of functions : Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two functions of non empty sets A, B, C then $gof : A \rightarrow C$ is called composition of f and g defined as $gof(x) = g\{f(x)\}\ \forall\, x \in A$
- If $f : A \rightarrow B$ and I_A and I_B are identity functions on A and B respectively then $foI_A = I_Bof = f$
- If $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$. Then, $fog \neq gof$ and $fo(goh) = (fog)oh$.
- If $n(A) = m$ and $n(B) = n$, then
  - (i) Number of one-one functions from A to $B = {}^nP_m$, where $n \geq m$
  - (ii) Number of onto functions (surjections) from A to B
    $= \sum_{r=1}^{n} (-1)^{n-r}\, {}^nC_r\, r^m$ provided $m \geq n$.
- Binary operation : A binary operation $*$ on set A is a function $* : A \times A \rightarrow A$. We denote $*$ on (a, b) by $a * b$.
  - $a * b = b * a\ \forall\, a, b \in A$ (Commutative law)
  - $(a * b) * c = a * (b * c)$ (Associative law)
  - $e * a = a = a * e$ (Law of identity)
  - $a * b = e = b * a$ (Law of inverse)
  - $a * b = a * c \Rightarrow b = c$
  - $b * a = c * a \Rightarrow b = c$

WORK IT OUT

VERY SHORT ANSWER TYPE

1. The binary operation $*$: $R \times R \to R$ is defined as $a * b = 2a + b$. Find $(2 * 3) * 4$.

2. Let $A = \{3, 4, 5\}$ and relation R on set A is defined as $R = \{(a, b) \in A \times A : a - b = 10\}$. Is relation an empty relation ?

3. If functions f and g are given by $f = \{(1, 2), (3, 5), (4, 1), (2, 6)\}$ and $g = \{(2, 6), (5, 4), (1, 3), (6, 1)\}$, find the range of f and g and write down the functions fog and gof.

4. Test the functions for one-one and onto :
(i) $f : R \to R$ defined by $f(x) = x^2 + 3; \forall x \in R$

5. $*$ is a binary operation defined on Z. Find the binary operation $a * b = a + b - 4 \;\forall\; a, b \in Z$ is commutative or not.

SHORT ANSWER TYPE

6. Let $f : R - \left\{-\frac{4}{3}\right\} \to R - \left\{\frac{4}{3}\right\}$ be a function defined as $f(x) = \frac{4x}{3x+4}$, find f^{-1} : Range of $f \to R - \left\{-\frac{4}{3}\right\}$.

7. If $f : R \to R : f(x) = 2x + 3$ then find f^{-1}.

8. Consider $f : R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of f given by $f^{-1}(y) = \sqrt{y-4}$, where R_+ is the set of all non-negative real numbers.

9. Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let R_1 and R_2 be relations in X given by $R_1 = \{(x, y) : x - y$ is divisible by 3$\}$ and $R_2 = \{(x, y) : \{x, y\} \subset \{1, 4, 7\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \in \{3, 6, 9\}$ where '$\subset$' means subset of. Show that $R_1 = R_2$.

10. If $f, g : R \to R$ are defined respectively by $f(x) = x^2 + 3x + 1$, $g(x) = 2x - 3$, find
(i) fof (ii) gog

LONG ANSWER TYPE-I

11. Let f and g be two real functions as $f(x) = 2x - 3$; $g(x) = \frac{3+x}{2}$. Find fog and gof. Can you say one is inverse of the other?

12. Let $*$ be a binary operation on Q, defined by $a * b = \frac{3ab}{5}$. Show that $*$ is commutative as well as associative. Also, find its identity, if it exists.

13. Prove that : The inverse of an equivalence relation is also an equivalence relation.

14. If the function $f : R \to R$ be given by $f(x) = x^2 + 2$ and $g : R \to R$ be given by $g(x) = \frac{x}{x-1}$, $x \neq 1$, find fog and gof and hence find $(fog)(2)$ and $(gof)(-3)$.

15. Let $f : \{1, 2, 3\} \to \{a, b, c\}$ be one-one and onto function given by $f(1) = a$, $f(2) = b$ and $f(3) = c$. Show that there exists a function $g : \{a, b, c\} \to \{1, 2, 3\}$ such that $gof = I_X$ and $fog = I_Y$ where $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$.

LONG ANSWER TYPE-II

16. Let P be the set of all the points in a plane and the relation R in set P be defined as $R = \{(A, B) \in P \times P \mid$ distance between points A and B is less than 3 units$\}$. Show that the relation R is not an equivalence relation.

17. Consider the binary operations $*$: $R \times R \to R$ and $o : R \times R \to R$ defined as $a * b = |a - b|$ and $a \; o \; b = a$, $\forall\; a, b \in R$. Show that $*$ is commutative but not associative, o is associative but not commutative.

18. Let $A = R - \{2\}$ and $B = R - \{1\}$. If $f : A \to B$ is a mapping defined by $f(x) = \frac{x-1}{x-2}$, show that f is bijective.

19. Let $f : N \to Y : f(x) = 4x^2 + 12x + 15$ and Y = range (f). Show that f is invertible and find f^{-1}.

20. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and R be a relation in $A \times A$, defined by $(a, b) \; R \; (c, d) \Leftrightarrow a + d = b + c$ for all (a, b) and $(c, d) \in A \times A$. Prove that R is an equivalence relation. Also obtain the equivalence class determined by $(2, 5)$.

SOLUTIONS

1. Given, $a * b = 2a + b \Rightarrow (2 * 3) * 4 = (4 + 3) * 4 = 7 * 4 = 14 + 4 = 18$

2. We notice for no value of $a, b \in A$, $a - b = 10$. Hence, $(a, b) \notin R$ for $a, b \in A$. Hence, R is an empty relation.

3. Range $f = \{1, 2, 5, 6\}$; Range $g = \{1, 3, 4, 6\}$
$fog\,(2) = f(g(2)) = f(6)$, not defined
$\therefore$ fog is not defined.
Now, $gof\,(1) = g(f(1)) = g(2) = 6$
$gof\,(3) = g(f(3)) = g(5) = 4$
$gof\,(4) = g(f(4)) = g(1) = 3$
$gof\,(2) = g(f(2)) = g(6) = 1$
$\therefore$ $gof = \{(1, 6), (3, 4), (4, 3), (2, 1)\}$

4. (i) Given, $f: R \to R$ defined by $f(x) = x^2 + 3\ \forall\ x \in R$
Clearly $-1, 1 \in R$ and $f(-1) = 4$ and $f(1) = 4$
Thus two different elements -1 and 1 of domain have same image under f and hence f is not one-one.
Since, $f(x) = x^2 + 3 \geq 3$ $[\because x^2 \geq 0]$
Hence, range $f = [3, \infty) = \{y : 3 \leq y < \infty\} \neq$ codomain R
Hence, f is not onto
Thus f is neither one-one nor onto.

5. $a * b = a + b - 4 = b + a - 4 = b * a\ \forall\ a, b \in Z$
Hence, the operation $*$ is commutative.

6. Let $y = f(x)$, Then, $y = \dfrac{4x}{3x+4} \Rightarrow 3xy + 4y = 4x$
$\Rightarrow x(4 - 3y) = 4y$

$$\Rightarrow x = \frac{4y}{4-3y}$$

$$\therefore\ f^{-1}(y) = \frac{4y}{4-3y} \text{ or } f^{-1}(x) = \frac{4x}{4-3x}$$

7. Let $y = f(x)$ Then, $y = 2x + 3$

$$\Rightarrow x = \frac{1}{2}(y - 3)$$

$$\therefore\ f^{-1}(y) = \frac{1}{2}(y - 3) \qquad [\because y = f(x) \Rightarrow x = f^{-1}(y)]$$

Thus, we define $f^{-1} : R \to R\ ; f^{-1}(y) = \dfrac{1}{2}(y - 3)$

8. For one-one : Let $x_1, x_2 \in R_+$ and consider $f(x_1) = f(x_2)$
$\Rightarrow x_1^2 + 4 = x_2^2 + 4$
$\Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = x_2 (\because x_1, x_2 \in R_+)$. So, f is one-one.
For onto : Let $y \in [4, \infty)$ and $x \in R_+$ such that $f(x) = y$
$\Rightarrow x^2 + 4 = y \Rightarrow x = \sqrt{y-4} \in R_+$. Hence, f is onto.
Since, f is one-one and onto.
$\therefore$ f is invertible, with $x = \sqrt{y-4}$ or $f^{-1}(y) = \sqrt{y-4}$.

9. Given, $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
Let $A_1 = \{1, 4, 7\}$, $A_2 = \{2, 5, 8\}$, $A_3 = \{3, 6, 9\}$
Clearly difference of any two elements of sets A_1 or A_2 or A_3 is a multiple of 3
Now $(x, y) \in R_1 \Leftrightarrow x - y$ is a multiple of 3
$\Leftrightarrow$ (x, y) belong to the same set A_1 or A_2 or A_3
$\Leftrightarrow$ $\{x, y\} \subset A_1$ or $\{x, y\} \subset A_2$ or $\{x, y\} \subset A_3$
$\Leftrightarrow$ $(x, y) \in R_2$
Hence, $R_1 = R_2$

10. Given, $f(x) = x^2 + 3x + 1$ and $g(x) = 2x - 3$
(i) For any $x \in R$, we have
$(fof)(x) = f(f(x)) = f(x^2 + 3x + 1)$
$= (x^2 + 3x + 1)^2 + 3(x^2 + 3x + 1) + 1$
$= x^4 + 9x^2 + 1^2 + 6x^3 + 2x^2 + 6x + 3(x^2 + 3x + 1) + 1$
$= x^4 + 6x^3 + 14x^2 + 15x + 5$
Hence, $fof : R \to R$ is defined by
$(fof)(x) = x^4 + 6x^3 + 14x^2 + 15x + 5$, for all $x \in R$
(ii) For any $x \in R$, we have
$(gog)(x) = g(g(x)) = g(2x - 3) = 2(2x - 3) - 3 = 4x - 9$
Hence, $gog : R \to R$ is defined by $(gog)(x) = 4x - 9$, for all $x \in R$

11. $fog(x) = f(g(x)) = f\left(\dfrac{3+x}{2}\right) = 2\left(\dfrac{3+x}{2}\right) - 3 = 3 + x - 3 = x$

$$gof(x) = g(f(x)) = g(2x - 3) = \frac{3+2x-3}{2} = \frac{2x}{2} = x$$

As fog and gof are identity functions, so one is inverse of the other.

12. For commutative : Let $a, b \in Q \Rightarrow a * b = \dfrac{3ab}{5}$ and $b * a = \dfrac{3ba}{5}$

As $\dfrac{3ab}{5} = \dfrac{3ba}{5} \Rightarrow a * b = b * a$. Hence, $*$ is commutative.

For associative : Let $a, b, c \in Q \Rightarrow (a * b) * c = \left(\dfrac{3ab}{5}\right) * c$

$$= \frac{3\left(\dfrac{3ab}{5}\right)c}{5} = \frac{9abc}{25}. \qquad \text{...(i)}$$

$$a * (b * c) = a * \frac{3bc}{5} = \frac{3a\left(\dfrac{3bc}{5}\right)}{5} = \frac{9abc}{25} \qquad \text{...(ii)}$$

From (i) and (ii), $(a * b) * c = a * (b * c)$
Hence, $*$ is associative
For identity : If $e \in Q$ is an identity element for binary operation $*$, then for $a \in Q$,

$$a * e = e * a = a \Rightarrow \frac{3ae}{5} = \frac{3ea}{5} = a \Rightarrow e = \frac{5}{3}.$$

13. Let R be an equivalence relation on a set A. Then R is reflexive, symmetric and transitive.
(a) For reflexivity, let $(x, x) \in R,\ \forall\ x \in A$
$\Rightarrow$ $(x, x) \in R^{-1} \Rightarrow R^{-1}$ is reflexive
(b) $(x, y) \in R^{-1} \Rightarrow (y, x) \in R \Rightarrow (x, y) \in R \Rightarrow (y, x) \in R^{-1}$
$[\because R$ is symmetric$]$
$\therefore$ R^{-1} is symmetric
(c) $(x, y) \in R^{-1}$ and $(y, z) \in R^{-1}$
$\Rightarrow$ $(y, x) \in R$ and $(z, y) \in R$
$\Rightarrow$ $(z, y) \in R$ and $(y, x) \in R$
$\Rightarrow$ $(z, x) \in R$ $(\because R$ is transitive$)$
$\Rightarrow$ $(x, z) \in R^{-1}$
Thus, R^{-1} is transitive
Hence, R^{-1} is an equivalence relation on A.

14. Given, $f(x) = x^2 + 2$ and $g(x) = \dfrac{x}{x-1}$, $x \neq 1$

$(fog)(x) = f(g(x)) = f\left(\dfrac{x}{x-1}\right) = \left(\dfrac{x}{x-1}\right)^2 + 2$

$(fog)(x) =$

$$\frac{x^2+2(x-1)^2}{(x-1)^2} = \frac{x^2+2x^2-4x+2}{(x-1)^2} = \frac{3x^2-4x+2}{(x-1)^2} \quad ...\text{(i)}$$

$$(gof)(x) = g(f(x)) = g(x^2+2) = \frac{x^2+2}{x^2+2-1} = \frac{x^2+2}{x^2+1} \quad ...\text{(ii)}$$

$$(fog)(2) = \frac{3(2)^2-4\times2+2}{(2-1)^2} = \frac{12-8+2}{1} = 6 \qquad \text{[from (i)]}$$

$$(gof)(-3) = \frac{(-3)^2+2}{(-3)^2+1} = \frac{9+2}{9+1} = \frac{11}{10} \qquad \text{[from (ii)]}$$

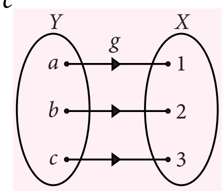
15. Let $g : \{a, b, c\} \to \{1, 2, 3\}$ be defined by

$g(a) = 1$, $g(b) = 2$ and $g(c) = 3$

i.e., $g = \{(a, 1), (b, 2)\ (c, 3)\}$

Also, $f(1) = a$, $f(2) = b$ and $f(3) = c$

i.e., $f = \{(1, a), (2, b), (3, c)\}$



Now, $gof(1) = g[f(1)] = g(a) = 1$

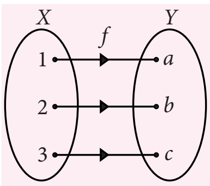
$gof(2) = g[f(2)] = g(b) = 2$

and $gof(3) = g[f(3)] = g(c) = 3$

Thus $gof : X \to X$ defined by

$gof = \{(1, 1), (2, 2), (3, 3)\}$ = the identity function on X

Hence $gof = I_X$



Again, $fog(a) = f[g(a)] = f(1) = a$

$fog(b) = f[g(b)] = f(2) = b$

and $fog(c) = f[g(c)] = f(3) = c$

Thus $fog : Y \to Y$ defined by

$fog = \{(a, a), (b, b), (c, c)\}$ = the identity function on Y

Hence $fog = I_Y$

16. Given, $R = \{(A, B) \in P \times P \mid$ distance between points A and B is less than 3 units$\}$

For reflexivity : $(A, A) \in R$ is true as distance between points A and A is 0, which is less than 3 units for all $A \in P$. Hence, R is reflexive.

For symmetry : Let $A, B \in P$

$(A, B) \in R \Rightarrow$ distance between points A and B is less than 3 units.

$\Rightarrow$ Distance between B and A is less than 3 units.

So, $(B, A) \in R$

Hence, R is symmetric.

For transitivity : Let points A, B and C are collinear. B is mid-point of AC such that distance between A and B is 2 units and between B and C is also 2 units, *i.e.*, $(A, B) \in R$ and $(B, C) \in R$, we notice distance between A and C is 4 units $\Rightarrow (A, C) \notin R$. Hence, R is not transitive.

Hence, R is not an equivalence relation.

17. Consider binary operation $a * b = |a - b|$

$a * b = |a - b|$ and $b * a = |b - a| = |-(a - b)| = |a - b|$

As $a * b = b * a\ \forall\ a, b \in R$. Hence, $*$ is commutative.

Let $a = 2$, $b = 3$ and $c = 4$

$(a * b) * c = (2 * 3) * 4 = |2 - 3| * 4 = |1 - 4| = 3$

$a * (b * c) = 2 * (3 * 4) = 2 * |3 - 4| = 2 * 1 = |2 - 1| = 1$

As $(a * b) * c \neq a * (b * c)$. Hence, $*$ is not associative.

Consider binary operation $a\ o\ b = a\ \forall\ a, b \in R$

$aob = a$ and $boa = b$

As $aob \neq boa$. Hence, o is not commutative.

Also consider $(aob)oc = aoc = a$ and $ao(boc) = aob = a$

As $(aob)oc = ao(boc)\ \forall\ a, b, c \in R$. Hence, o is associative.

18. Consider function $f(x) = \dfrac{x-1}{x-2}$

For one-one : Let $x, y \in A$ and consider $f(x) = f(y)$

$\Rightarrow \dfrac{x-1}{x-2} = \dfrac{y-1}{y-2}$

$\Rightarrow (x - 1)(y - 2) = (x - 2)(y - 1)$

$\Rightarrow xy - y - 2x + 2 = xy - x - 2y + 2$

$\Rightarrow x = y$

Thus, $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$

So, f is one-one.

For onto : Let y be an arbitrary element of B. Then,

$f(x) = y \Rightarrow \dfrac{x-1}{x-2} = y \Rightarrow (x - 1) = y(x - 2)$

$\Rightarrow x = \dfrac{1-2y}{1-y}$

Clearly, $x = \dfrac{1-2y}{1-y}$ is a real number for all $y \neq 1$.

Also, $\dfrac{1-2y}{1-y} \neq 2$ for any y, for, if we take $\dfrac{1-2y}{1-y} = 2$, then we get $1 = 2$, which is wrong.

So, f is onto. Hence, f is a bijective map.

MPP-1 CLASS XII ANSWER KEY

1. (d)	**2.** (b)	**3.** (b)	**4.** (a)	**5.** (a)
6. (a)	**7.** (a,b,c)	**8.** (a, b, c, d)		**9.** (d)
10. (a, c)	**11.** (a, b)	**12.** (a, b)	**13.** (a, b, c)	
14. (a)	**15.** (b)	**16.** (c)	**17.** (1)	**18.** (9)
19. (0)	**20.** (8)			

19. For one-one : Let $x_1, x_2 \in N$ and consider,

$f(x_1) = f(x_2)$

$\Rightarrow 4x_1^2 + 12x_1 + 15 = 4x_2^2 + 12x_2 + 15$

$\Rightarrow\ 4(x_1^2 - x_2^2) + 12(x_1 - x_2) = 0$

$\Rightarrow\ (x_1^2 - x_2^2) + 3(x_1 - x_2) = 0$

$\Rightarrow\ (x_1 - x_2)(x_1 + x_2 + 3) = 0$

$\Rightarrow\ x_1 - x_2 = 0 \qquad [\because x_1 + x_2 + 3 \neq 0]$

$\Rightarrow\ x_1 = x_2$. So, f is one-one

For onto : range $(f) = Y$. So, f is onto.

Thus, f is one-one and onto. $\therefore$ f is invertible.

Let $y \in Y$. Then, f being onto, there exists x such that $y = f(x)$

Now, $y = f(x) \Rightarrow y = 4x^2 + 12x + 15$

$\Rightarrow\ y = (2x + 3)^2 + 6$

$\Rightarrow\ (2x + 3) = \sqrt{y-6} \quad \Rightarrow x = \frac{1}{2}\left(\sqrt{y-6}-3\right)$

$\Rightarrow\ f^{-1}(y) = \frac{1}{2}\left(\sqrt{y-6}-3\right)$

Thus, we define :

$f^{-1} : Y \to N : f^{-1}(y) = \frac{1}{2}\left(\sqrt{y-6}-3\right)$

20. (i) For reflexivity : Let $(a, b) \in A \times A$. Then,

$(a, b) \in A \times A \Rightarrow a, b \in A$

$\Rightarrow\ a + b = b + a$

$\Rightarrow\ (a, b)\ R\ (a, b)$

$\therefore\ R$ is reflexive.

(ii) For symmetry : Let $(a, b)\ R\ (c, d)$. Then,

$(a, b)\ R\ (c, d) \Rightarrow a + d = b + c0$

$\Rightarrow\ c + b = d + a$

$\Rightarrow\ (c, d)\ R\ (a, b)$

$\therefore\ R$ is symmetric.

(iii) For transitivity : Let $(a, b)\ R\ (c, d)\ R\ (e, f)$. Then,

$(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$

$\Rightarrow\ a + d = b + c$ and $c + f = d + e$

$\Rightarrow\ a + d + c + f = b + c + d + e$

$\Rightarrow\ a + f = b + e$

$\Rightarrow\ (a, b)\ R\ (e, f)$

$\therefore\ R$ is transitive.

Thus, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

$[(2, 5)] = \{(a, b) : (2, 5)\ R\ (a, b)\}$

$= \{(a, b) : 2 + b = 5 + a\} = \{(a, b) : b - a = 3\}$

$= \{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$.

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MPP-1 MONTHLY Practice Problems

Class XII

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Relations and Functions

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. The inverse of the function $f(x) = \log_2(x+\sqrt{x^2+1})$ is

(a) $2^x + 2^{-x}$ (b) $\frac{2^x+2^{-x}}{2}$

(c) $\frac{2^{-x}-2^x}{2}$ (d) $\frac{2^x-2^{-x}}{2}$

2. Let R be the relation on the set $A = \{1, 2, 3, 4\}$ given by $R = \{(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3,2)\}$. Then,

(a) R is reflexive and symmetric but not transitive
(b) R is reflexive and transitive but not symmetric
(c) R is symmetric and transitive but not reflexive
(d) R is an equivalence relation

3. If the function $f: (-\infty, \infty) \to B$ defined by $f(x) = -x^2 + 6x - 8$ is bijective, then B is equal to

(a) $[1, \infty)$ (b) $(-\infty, 1]$
(c) $(-\infty, \infty)$ (d) None of these

4. If $f(x) = \frac{x-1}{x+1}$, then $f(f(ax))$ in terms of $f(x)$ is equal to

(a) $\frac{f(x)-1}{a(f(x)+1)}$ (b) $\frac{f(x)+1}{a(f(x)-1)}$

(c) $\frac{f(x)-1}{a(f(x)-1)}$ (d) $\frac{f(x)+1}{a(f(x)+1)}$

5. If $g(f(x)) = |\sin x|$ and $f(g(x)) = (\sin\sqrt{x})^2$, then

(a) $f(x)=\sin^2 x, g(x)=\sqrt{x}$
(b) $f(x) = \sin x, g(x) = |x|$
(c) $f(x)=x^2, g(x)=\sin\sqrt{x}$
(d) f and g cannot be determined.

6. Let $f: R-\left\{\frac{3}{5}\right\} \to R$ be defined by $f(x)=\frac{3x+2}{5x-3}$. Then

(a) $f^{-1}(x) = x$ (b) $f^{-1}(x) = -f(x)$
(c) $fof(x) = -x$ (d) $f^{-1}(x)=\frac{1}{19}f(x)$

One or More Than One Option(s) Correct Type

7. Let $f(x)$ be a real valued function such that $f(0)=\frac{1}{2}$ and $f(x + y) = f(x) f(a - y) + f(y) f(a - x)\ \forall\ x, y \in R$, then for some real a

(a) $f(x)$ is a periodic function
(b) $f(x)$ is a constant function
(c) $f(x)=\frac{1}{2}$
(d) $f(x)=\frac{\cos x}{2}$

8. Let $f: R \to R$ be a function defined by $f(x+1)=\frac{f(x)-5}{f(x)-3}\ \forall\ x \in R$. Then, which of the following statements is/are true?

(a) $f(2008) = f(2004)$ (b) $f(2006) = f(2010)$
(c) $f(2006) = f(2002)$ (d) $f(2006) = f(2018)$

9. If $f(x)=\sqrt{3|x|-x-2}$ and $g(x) = \sin x$, then domain of definition of $fog(x)$ is

(a) $\left\{2n\pi+\frac{\pi}{2}\right\}_{n\in I}$

(b) $\bigcup_{n\in I}\left(2n\pi+\frac{7\pi}{6}, 2n\pi+\frac{11\pi}{6}\right)$

(c) $\left\{2n\pi+\frac{7\pi}{6}\right\}_{n\in I}$

(d) $\left(2n\pi+\frac{7\pi}{6}, 2n\pi+\frac{11\pi}{6}\right) \bigcup\limits_{n, m\in I} \left(2m\pi+\frac{\pi}{2}\right)$

10. The function $f: R \to R$ defined by $f(x) = 2^x + 2^{|x|}$ is
(a) one-one (b) onto
(c) into (d) many-one

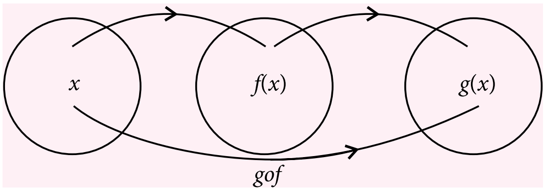
11. The distinct linear functions which map $[-1, 1]$ onto $[0, 2]$ are
(a) $f(x) = x + 1$ (b) $f(x) = -x + 1$
(c) $f(x) = -x - 1$ (d) $f(x) = 2x + 2$

12. Let $f(x + y) + f(x - y) = 2f(x) f(y)$, $\forall\ x, y \in R$ and $f(0) = k$, then
(a) f is even, if $k = 1$ (b) f is odd, if $k = 0$
(c) f is always odd
(d) f is neither even nor odd for any value of k

13. Let R be the relation from $A = \{2, 3, 4, 5\}$ to $B = \{3, 6, 7, 10\}$ defined by 'x divides y', then subsets of R^{-1} are
(a) $\{(6, 2), (3, 3)\}$
(b) $\{(6, 2), (3, 3), (10, 5)\}$
(c) $\{(6, 2), (10, 2), (3, 3), (6, 3)\}$
(d) $\{(3, 2), (7, 3), (10, 5)\}$

Comprehension Type

Let A, B and C be three non-void sets and let $f: A \to B$, $g: B \to C$ be two functions. Since f is a function from A to B, therefore for each $x \in A$ there exists a unique element $f(x) \in B$. Again, since g is a function from B to C therefore corresponding to $f(x) \in$ B there exists a unique element $g(f(x)) \in C$. Thus for each $x \in A$ there exists a unique element $g(f(x)) \in C$.



Let $f: A \to$ B and $g: B \to C$ be two functions, then a function $gof: A \to C$ defined by
$$(gof)(x) = g(f(x)), \text{ for all } x \in A.$$

14. If $f(x) = \sin^2 x + \sin^2(x + \pi/3) + \cos x \cos(x + \pi/3)$ and $g(5/4) = 1$, then $gof(x) =$
(a) 1 (b) 0
(c) $\sin x$ (d) none of these

15. Given $f(x) = \log\left(\frac{1+x}{1-x}\right)$ and $g(x) = \frac{3x+x^3}{1+3x^2}$, then $fog(x)$
(a) $-f(x)$ (b) $3f(x)$
(c) $[f(x)]^3$ (d) none of these

Matrix Match Type

16. Match the following :

	Column I		Column II
P.	If $F: [1, \infty) \to [2, \infty)$ is given by $f(x) = x + \frac{1}{x}$, then $f^{-1}(2)$ equals	1.	3
Q.	If function $f: R \to R$ defined as $f(x) = 2x - 3$ then $f^{-1}(3)$ is	2.	$-1/2$
R.	Let $f: N \to R$ be a function defined as $f(x) = 4x^2 + 12x + 15$ is invertible then $f^{-1}(10)$ is	3.	1

	P	Q	R
(a)	2	1	3
(b)	3	2	1
(c)	3	1	2
(d)	1	2	3

Integer Answer Type

17. The minimum value of $2^{(x^2-3)^3+27}$ is

18. Let $f: R \to R$ be defined as $f(x) = \begin{cases} 2x, & \text{if } x > 3 \\ x^2, & \text{if } 1 < x \le 3 \\ 3x, & \text{if } x \le 1 \end{cases}$

Then, $f(-1) + f(2) + f(4)$ is

19. Let $*$ be a binary operation defined on set $Q - \{1\}$ by the rule $a * b = a + b - ab$. Then, the identity element for $*$ is

20. The period of the function $f(x)$ which satisfies the relation $f(x) + f(x + 4) = f(x + 2) + f(x + 6)$ is

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Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ......
No. of questions correct ......
Marks scored in percentage ......

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

ANDHRA PRADESH TOPS LIST OF APPLICANTS

OVER 2.23 LAKH CANDIDATES SET TO APPEAR FOR VITEEE-2017

VELLORE: VIT University has yet again created history with a whopping 2.23 lakh set to appear for VIT Engineering Entrance Examination 2017.

This year, the university has registered 2,23,081 students for VITEEE, whereas the number was 2,12,238 last year. In addition, this year the university also received 10,843 more applications, which show the brand's reputation among students across the nation.

Announcing the results at a press conference in VIT University Founder & Chancellor, Dr. G. Viswanathan said that the university's record placement this year and its thrust to create innovation in academics has propelled the increase in patronage among students, especially from the Northern part of India and non-resident Indians.

Among the prominent states in India, Andhra Pradesh tops the chart with 34,068 registrations, which includes 25,011 male applicants, 9,054 female applicants. Second on the list is Uttar Pradesh with 23,360 registrations followed by Telangana with 19,847, Maharashtra with 19,684 and Rajasthan with 16,304 and Tamil Nadu with 16,173 registration.

As per the centre-wise calculations, while Hyderabad registered 16,856, Delhi has registered 15,079 candidates, Vijayawada has registered 13,209, Kota has registered 8,877 candidates and Chennai has registered 7,687. Patna has registered 7,321 candidates and Vellore 2,789.

About VIT Engineering Entrance Examination (VITEEE)

VIT Engineering Entrance Examination (VITEEE) is to be held as Computer-Based Test from 5th April to 16th April 2017, in 119 cities, with 167 centres across India, including Dubai, Kuwait and Muscat for admission to B.Tech. programmes offered by VIT University in Vellore, Chennai, Bhopal (MP) and Amaravathi (AP).

Dr. G. Viswanathan also said that the university would announce VITEEE results on or before 24th April (tentatively) in www.vit.ac.in

The counseling for admissions will begin from 10th to 13th May, 2017, with each day divided by the ranks in ascending order. The counseling for those with ranks upto 8,000 will be held on 10th May followed by those with ranks upto 12,000 on the 11th, Ranks upto 16,000 will sit for counseling on 12th May and those with ranks upto 20,000 will sit on 13th May 2017.

Scholarship under GV School Development programme

To help deserving students get high quality education at VIT, the university has instituted special Scholarships. Dr. Viswanathan said that Central and state board toppers would get 100 percent fee waiver for all four years.

Performance	Scholarship*
Toppers of each State Board and Central Board	100% Tuition fee waiver for all the four years.
VITEEE rank holders of 1 to 50	75% Tuition fee waiver for all the four years.
VITEEE rank holders of 51 to 100	50% Tuition fee waiver for all the four years.
VITEEE rank holders of 101 to 1000	25% Tuition fee waiver for all the four years.

*Terms and Conditions Apply

STARS (Supporting the advancement of Rural Students)

To encourage more Tamil Nadu students to study in VIT, the university has been offering 100 percent waiver in tuition fees and exemption from hostel fee for two plus two toppers from each district in the state under the Supporting the Advancement of Rural Students (STARS) scheme.

"This effort has been made to ensure that high scorers from each district who have poor economic background get their due and are given the best of educations", said the Chancellor.

Speaking about the need for reforms in higher education, Dr. G. Viswanathan.

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